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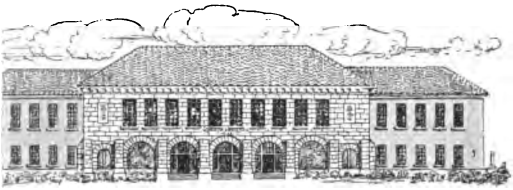
NEWELL AND HARPER



HIGH SCHOOL ALGEBRA

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CHICAGO NEW YORK

ROW, PETERSON AND COMPANY

PREFACE

That there are some good text-books in beginning algebra may be freely granted, and yet those who are familiar with the work of first-year high-school students know that many fail in that subject. Several conditions contribute to these failures, but there is no doubt that the text-books in use are partly responsible. Extensive experience in teaching the subject and in supervising the work of other teachers has convinced the authors that a book can be made that is distinctly better in a number of respects than any beginning text heretofore published.

Accordingly, the aim has been to make a book in which the explanations are so clear and the exercises and problems so well graded that a student of average ability, ambition and industry, could by private study work through it with little or no assistance from a teacher.

With this idea in mind the authors during some years kept notes of the more significant experiences in teaching and supervising the subject. Gradually a manuscript was developed, and the material, on mimeographed sheets, used as a regular text in several classes. While it was being tested in this way the results were checked, exercises and problems re-arranged as seemed desirable, and the whole work revised in the light of the response of the students.

Following are some of the special features of the book:

1. Explanations and illustrations are simple and clear.

2. The fundamental algebraic ideas are based upon and evolved from the arithmetical ideas already familiar.

3. In a similar way, every important new principle is approached by the use of ideas already well known to the learner. The students never come to a blank wall.

4. Division and factoring are presented concurrently with multiplication so that when factoring is taken up as a special subject the pupils are already familiar with many of the processes involved.

5. The exercises begin with the simplest possible examples and, as the difficulty is gradually increased, hints and suggestions are given that enable the student to make a maximum of progress with a minimum of help. Practically all of the examples in these exercises have been worked by a large number of first-year students.

6. Generally the examples at the beginning of a set of exercises are so simple that they may be taken up orally during the class period, so that the student can go on with the work between recitations with entire satisfaction.

7. There is an unusually large number of carefully graded verbal problems, that is, problems to be translated into equations and solved. These include every possible variety within the range of the ability of first-year students. Geometry supplies a great variety of examples, and this work represents more than 20% of the whole course.

8. Special attention has been given to the use of the graph and it is believed that this part of the course is more than usually interesting and satisfactory to the students.

9. The solution of the simpler forms of the quadratic equation is introduced early and thus provides for its application to a greater variety of types of problems.

10. Provision is made for frequent and very comprehensive reviews. There are two sets of review exercises in Chapter I, occupying six pages. In Chapters I to X there are eight of these review lists. Chapter XI is a cumulative review. This material may be used for supplementary work as the class advances during the year, for written tests at any stage of the work, or as a review of the whole subject at the end of the year. It could also be used as

the basis for a general review at any time, if a student desires to take an examination, and it should be used as an introduction to the "Second Course."

The "Second Course in Algebra," beginning page 241, is designed for use in the second or third year of the high-school course.

In common practice the mathematics courses in high-schools require a year of geometry following the first year algebra, but this course does not assume a knowledge of geometry; it may follow in a continuous study of the subject through a year and a half or two years, as required in some schools.

Chapter XI provides for a thorough review of the essentials taught in the first year course. The extent to which this chapter will be used is left to the discretion of the teacher. Many of the examples should be worked orally during the first few days of the term. In this way the pupils revive their knowledge of algebraic facts and principles, and renew their habits of thinking in algebraic terms; while the teacher has an opportunity to study the preparation and power of his pupils and to decide where his teaching should begin.

The work follows the same plan as the first year course:

1. There is the same skill in introducing each new topic, the same care in grading examples and problems, the same use of helpful hints, frequent and ample reviews, and extensive use of verbal problems, which challenge attention and foster interest throughout the course.

2. Logarithms are presented early and their application made attractive, a feature of great practical value.

3. Teachers will find it rarely necessary to use all the examples in one exercise to ensure mastery of rules and principles.

THE AUTHORS.

SUGGESTIONS TO THE TEACHER

1. Proceed slowly at first. Assign short lessons so that every topic of Chapters I and II can be thoroughly discussed during the class period and be well mastered by all.

2. Review all essential topics again and again—not only such as come early in the course but in all the work of the year. Use freely the lists given for this purpose whenever pupils seem uncertain as to important processes.

3. No class will need to work all of the problems of the text, but nothing should be omitted from Chapters I and II. Beginning with Chapter III, let the teacher select from the longer exercise lists such a number of examples as make a reasonable lesson, leaving the remainder for board work, for assignment to special pupils, or to be omitted entirely for the year. For instance, §73 has 48 examples. The assignment might well be 1-15 and ten odd numbered examples beginning with 17, making 25 in all.

4. Expect the average student to grow more and more self-reliant in grasping without assistance new explanations and illustrative examples, and in taking advantage of such hints and suggestions as are given in the exercise lists. Call on him to do his own explaining, not only of new varieties of problems as they appear in these lists but also of new processes. Encourage him to feel that the course would have no value, if there were no difficulties, but that it is expected that a student of fair ability and self-respect can overcome them all.

5. The largest possible number of the verbal problems should be worked and the graph used freely.

6. Algebra requires accuracy of work, and accuracy depends upon neatness and order. Prevent or check habits that come from lack of order in recording steps in the thinking process.

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A YEAR OF ALGEBRA

CHAPTER I

INTRODUCTION

1. Symbols of arithmetic. At the beginning of our study of arithmetic, we were taught to write numbers by the use of the symbols, 0, 1, 2, 3, 4, 5, etc.

Then we learned to use such numbers in the processes of addition, subtraction, multiplication and division.

Later, we spent much time finding relations and determining rules for solving many problems of every-day life. These problems involved common fractions, decimal fractions, percentage, denominate numbers, and other complicated forms.

We also learned to use some very convenient symbols to indicate operations and express relations, as $+$ for plus or addition, $-$ for minus or subtraction, $\times$ for times or multiplication, $\div$ or —— (the bar of the fraction) for divided by or division, and $=$ for equality.

The symbols of arithmetic as we use them today are relatively modern. The so-called Arabic Notation,—the use of the nine digits with 0 in the decimal or place system,—seems to have received its final development in India about the seventh century A. D. We know that this notation was not in very general use in Western Europe until about 1200 A. D. and that most of the symbols of operation have been adopted since that date.

The introduction of the Arabic Notation meant much to arithmetic, for there is always great gain in time and in ease of

operation when proper symbols and methods are employed in any line of scientific thinking. This is evident if the student attempts to solve the following problems, using both the modern Arabic Notation and the old Roman Notation,—the only notation known to the school boy of ancient times.

(1) Find the sum of twenty-four, one hundred thirty-six, and one thousand two hundred sixty-two.

Arabic	Roman
24	XXIV
136	CXXXVI
1262	MCCLXII

(2) Multiply two hundred forty-seven by thirty-five.

Arabic	Roman
247	CCXLVII
35	XXXV

The student may be interested in reading in some good history of mathematics how the school boy of ancient Rome solved these problems.

2. Symbols of algebra. Algebra is concerned essentially with symbols; it speaks in a language of symbols.

Arithmetic, by the use of a few symbols, enables us to solve quickly and accurately many practical problems.

Algebra, by a more extensive use of symbols, opens to us important fields of science and mathematics. It teaches us to choose symbols wisely and to use them skilfully.

Physics, chemistry, and other sciences, as well as trigonometry, surveying, and all the mathematics of the engineer, are closed to the student who does not have a working knowledge of the language of algebra.

In algebra as in arithmetic two kinds of symbols are used: (1) symbols that represent numbers, and (2) symbols that indicate relations and operations.

All the rules and symbols which were learned in arithmetic will be needed in algebra.

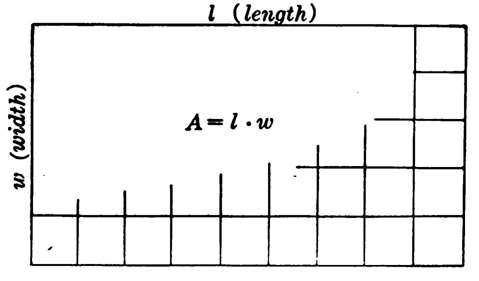
LITERAL NUMBERS

3. Letters used to represent numbers. The first thing to be learned in algebra is that numbers may be represented by letters as well as by Roman and Arabic numerals. We call such symbols for numbers, **literal numbers.**

The student knows the following:

Rule. *The area of a rectangle is the number of units in its length times the number of units in its width.*

This, in the short form called a **formula,** is $A = l \times w$ (also written $A = l \cdot w$) where A, l, and w are literal numbers and may have any number of different values.



In the above figure, l stands for the number of units in the length of the rectangle, w the number of units in its width, and A the number of square units in its area.

The student has used the formula $d = t \times r$ as applied to any moving object such as an automobile, where d is the number of miles covered on the trip, t the number of hours traveled, and r the rate, or number of miles made per hour.

Since r and t may each stand for any arithmetical number, $d = t \times r$ will be true for any trip. By the use of literal numbers a long rule is written in compact form.

It should be noticed that in the old Roman Notation each letter stood for a particular number as V for 5 and X for 10.

4. Symbols that indicate operations and relations. In arithmetic the sum of 5 and 3 is written $5+3$, the difference between 5 and 3 is $5-3$, the product of 5 and 3 is 5×3, the quotient of 5 divided by 3 is $5 \div 3$ or $\frac{5}{3}$, and the fact that two symbols for numbers are equal is indicated by the sign of equality, $=$, as $5+3=8$.

These symbols are used in algebra as in arithmetic.

5. Equality and its sign. The symbol of equality, $=$, is the most important symbol of algebra. **It is used to state the relation between two numbers that are known to have the same value.**

Notice the brevity of the algebraic statements for the following illustrative problems in mental arithmetic:

I. What number added to 8 gives 20?

In algebraic symbols this problem is, (1) $8+n=20$.
(Here n represents the number to be found.)

II. Twice what number minus 6 gives 12?

The algebraic problem is, (2) $2n-6=12$.

III. One number is 8 larger than another and their sum is 32. Find the numbers.

Let n be the larger number.

Then $n-8$ is the smaller number.

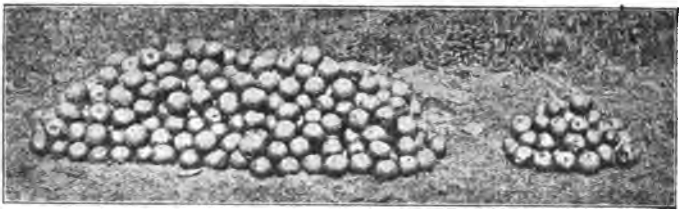
Then (3) $n+n-8=32$, or $2n-8=32$.

(1), (2), and (3) are known as **equations** and the value of n may be found by trial. Find the value of n in each.

Exercise 1

Find by inspection the value of the literal number in each of the following equations:

1. $n+5=7$.	**4.** $x-3=4$.	**7.** $2x+2=24$.
2. $n-3=7$.	**5.** $3n=30$.	**8.** $2a-5=15$.
3. $n+6=11$.	**6.** $3n+3=33$.	**9.** $5x+7=17$.



6. Addition and subtraction. If m is the number of apples in the first pile and n the number in the second pile, then the total number in both piles is $m + n$ and the difference between the piles is $m - n$.

If a third pile contains l apples, the sum of the apples in all three is $m + n + l$. $m + n - l$ is the sum of the apples in the first two minus the number in the third.

If there were two piles of b apples each, the total number in the two piles would be $b + b$ or $2b$, just as in arithmetic $10 + 10$ is 2 times 10 or two 10's.

7. Multiplication. The product of two literal numbers such as m and n may be written $m \times n$, $m \cdot n$, or better mn.

The symbols, $\times$ and $\cdot$ for "times," or "multiplied by," will be used only when necessary.

Since 2×3 gives the same result as 3×2, and $2 \times 3 \times 4$ is the same as $3 \times 2 \times 4$ or $4 \times 2 \times 3$, etc., we have the following:

Rule. *In finding the product of several numbers they may be taken in any order.*

Therefore $m \times n$ is the same as $n \times m$, or mn is the same as nm.

But it is usually more convenient to write the literal numbers of a product in their alphabetical order. The product of a, b, and c is written abc and read as a word is spelled "a-b-c."

Let the student be very sure to notice that the decimal or place system is not to be used with literal numbers,—that *abc* is a product. We have nothing like this in arithmetic when using the Arabic numerals, for 2 written by 3 in 23 is not 2×3.

It is also convenient when we wish to write the product of both Arabic and literal numbers to place the Arabic number, or the product of the Arabic numbers, first and then the product of the literal numbers.

For instance the product—

(1) of 2 and a is written $2a$ (to be read "two a"),

(2) of 2, a, and b is $2ab$,

(3) of $2a$ and $3b$ is $6ab$, and

(4) 2 times a times b times 3 times c is $6abc$.

8. Division. The quotient of m divided by n may be written $m \div n$, to be read "m divided by n," or $\frac{m}{n}$ to be read "the fraction m divided by n," or "m over n."

The fractional form is the more convenient one for literal numbers.

The various combinations of literal numbers such as, $a + b$, $a - b$, mn, $2abc$, $\frac{m}{n}$, etc., are called **algebraic expressions,** or simply, **expressions.**

Exercise 2. Forming Algebraic Expressions

1. What is the product of a and 3? of $2a$ and $5b$? of a, b, c, and 3?

2. What is the quotient when x is divided by 3? by a? by $2a$?

3. What is the quotient when $2a$ is divided by 3? by b? by $5b$?

4. What number is 3 more than 8? 3 more than x? Ans. $3 + x$.

5. What number is a more than x? $2a$ more than $3x$?

6. What number is 3 less than 8? 3 less than x? Ans. $x - 3$.

7. What number is a less than x? ab less than $2m$?

8. What is the sum of a, b, c, and d? of $2a$, $3b$, $4c$, and $5d$?

9. John has c cents and earns 5 more. How many cents has he?

10. If a boy walks 4 miles an hour, how many miles will he walk in 3 hours? in h hours? in $2h$ hours?

11. An auto travels 20 miles per hour. How many miles will it go in 4 hours? in h hours? in $2h$ hours?

12. James has c cents and spends 5 cents for a pencil. How many cents has he? Ans. $c - 5$.

13. John goes to the store with c cents in his pocket. He spends 5 cents for a pencil and b cents for a book. How many cents has he left. Ans. $c - 5 - b$.

14. If a man has d dollars on hand at the beginning of the month, earns e dollars during the month, and spends l dollars for living expenses, how much has he left at the end of the month?

15. If oranges cost 3 cents each, how many can be bought for 30 cents? for a cents? for $2a$ cents?

16. If a train runs 30 miles per hour, how many hours will be required to run 80 miles? to run m miles?

17. How many apples costing a cents each can be bought for b cents?

18. Write the number that is 3 more than 2 times n. Ans. $2n + 3$.

19. Write the number that is a more than 2 times b; n more than a times c; $2a$ more than x times y.

20. Write the number next higher than 8; the next higher than x. Ans. $x+1$.

Note. **Consecutive numbers** are those that follow in order in the number system, as 4, 5, 6, etc. **Consecutive even** numbers are even numbers in order, as 4, 6, 8, etc., and **consecutive odd** numbers are odd numbers in order as 5, 7, 9, etc.

21. Beginning with x write the consecutive numbers in order to $x+10$. The consecutive numbers down to $x-8$.

22. Beginning with 5 count by 3's to 32.

23. Beginning with x count by 3's to $x+18$.

24. Beginning with $n+1$ count by 2's to $n+15$.

25. If n is an odd number, write the next five odd numbers.

26. If n is an even number, write the next five even numbers.

27. Write three consecutive even numbers of which a is the least.

28. Write three consecutive odd numbers of which n is the second number.

29. Write three consecutive even numbers of which n is the largest.

Translate the following equations into problems of mental arithmetic and find the value of the literal number:

30. $2n+1=15$.

Translated gives, "twice what number added to 1 gives 15?"

31. $3n-5=25$.

32. $4n+4=20$.

33. $2n+12=52$.

34. $3n-10=26$.

35. $2n-15=21$.

36. $4n-10=30$.

Translate the following problems into equations and find the answer by inspection:

37. If 3 is added to 2 times a certain number, the sum is 21. What is the number?

Solution. Let n represent the number.
Then $2n + 3 = 21$. (By the conditions of the problem.)
Whence $n = 9$, the required number.

38. If 5 is added to 3 times a certain number, the sum is 35. What is the number?

39. If 3 is subtracted from 2 times a certain number, the remainder is 7. What is the number?

Hint. The equation is $2n - 3 = 7$.

40. Find two numbers such that the second is 3 more than the first and the sum is 35.

Solution. Let n represent the first number.
Then $n + 3$ represents the second number. (Why?)
Then $n + n + 3 = 35$. (Why?) Or $2n + 3 = 35$.
Whence $n = 16$, the first number.
and $n + 3 = 19$, the second number.

41. Find two numbers such that the second is 5 more than the first and their sum is 23.

42. Find two numbers such that the second is 4 less than the first and their sum is 14.

43. Find two numbers such that the second is 3 more than two times the first and their sum is 27.

Question: If n represents the first number, what will represent the second?

44. Find two consecutive numbers whose sum is 13.

45. Find three consecutive numbers whose sum is 42.

Hint. The equation is $n + n + 1 + n + 2 = 42$.

46. Find three consecutive numbers whose sum is 36.

47. A house and lot are worth \$8,000. If the house is worth 3 times as much as the lot, find the value of each.

Solution. Let v represent the number of dollars the lot is worth.
Then $3v =$ the number of dollars the house is worth.
Then $v + 3v = 8000$.
Whence $v = 2000$, the number of dollars the lot is worth, and $3v = 6000$, the number of dollars the house is worth.

Notice in the above solution that v and $3v$ are numbers and not actual values of property.

48. Robert bought two books that cost together 90 cents. If one of the books cost 2 times as much as the other, what did each book cost?

49. One of two numbers is 5 times the other and their sum is 48. What are the numbers?

50. The sum of three numbers is 66. The second is 2 times the first and the third is 3 times the first. What are the numbers?

51. James is 2 times as old as John and Henry is 3 times as old as John. The sum of the ages of the three boys is 30 years. Find the age of each.

52. A book and knife cost together 80 cents. Find the cost of each if the knife cost 20 cents more than the book.

9. Factors. The **factors** of a number are the numbers which, when multiplied together, give the original number.

For example, 2, 3, and 5 are factors of 30 since $2 \cdot 3 \cdot 5 = 30$. 3 and 10 are also factors of 30, and also 5 and 6.

Similarly, 3, a, and b are factors of $3ab$. Also $3a$ and b as well as $3b$ and a.

$6ab$ has the following pairs of factors: 2 and $3ab$, 3 and $2ab$, $2b$ and $3a$, $2a$ and $3b$, a and $6b$, b and $6a$, ab and 6.

Ex. **1.** Name as many pairs of factors of 24 as possible; of 40; of 64.

Note. One of the pairs of factors of 24 is 1 and 24.

Ex. **2.** Name as many pairs of factors of $12xy$ as possible; of $18xyz$; of 100; of $20abc$.

Ex. **3.** Name as many sets of three factors each of 108 as possible; of $18xyz$; of $24abc$.

10. Coefficient. One of the two factors of a number is called the **coefficient** of the other factor.

In the number $3ab$, 3 is the coefficient of ab, $3b$ of a, and $3a$ of b.

The **Arabic number** is known as the **Arabic** or **numerical coefficient** of the literal number of a product.

Note. When 1 is the numerical coefficient, it is omitted, as ab instead of $1ab$.

Ex. **1.** In $3abc$, what is the coefficient of bc? of ac? of $3c$? of $3a$? of $3ab$? of $3b$?

Ex. **2.** In $30xyz$, name the numerical coefficient; the coefficient of $5x$; of $3yz$; of $15xz$; of $2xy$; of $3z$; of $10x$.

11. Factoring completely. A number is said to be **completely factored** when no one of its factors can be separated into other factors without using fractions or other complicated forms.

When an arithmetical number is completely factored, these factors are called its **prime factors** and it is sometimes convenient to call such factors of an algebraic expression its prime factors.

Ex. **1.** What are the prime factors of 18? of 30? of 57? of 115?

Ex. **2.** Factor completely $12xy$; $18abc$; $24mrs$; $108axz$.

Ex. **3.** What are the prime factors of 1000? of 1444? of 2024?

12. Common factor. If two numbers have the same factor, that factor is called their **common factor.**

For example, 4 and 6 have the common factor, 2. $3a$ and $5a$ have the common factor, a. $3abx$ and $6bxy$ have several common factors, 3, b, and x.

The product of all the common prime factors of several numbers is called their **highest common factor.** (In arithmetic, **greatest common divisor.**)

For example, the highest common factor of $3abx$ and $6bxy$ is $3bx$.

Exercise 3

Find the greatest common divisor of the following:

1. 36 and 64.
2. 45 and 105.
3. 108 and 144.
4. 85 and 153.
5. 115 and 161.
6. 20, 30, 45, and 75.
7. 12, 18, 27, and 45.
8. 26, 65, 91, and 156.
9. 45, 75, 105, and 135.
10. 36, 48, 64, 108, and 144.

Name at sight a common factor for each of these:

11. $4ac$ and $5bc$.
12. $5xy$ and $10xz$.
13. $3abc$ and $5bcd$.
14. $18ab$, $24by$, and $32bz$.
15. $36xyz$, $54yzw$, and $72xyw$.
16. $27yzu$, $45xyu$, and $48xyzu$.

Find the H. C. F. (highest common factor) of these:

17. $8abc$, $12bc$, and $24ac$.
18. $24mnr$, $32nrs$, and $48mrs$.
19. $9abcd$, $12bcd$, $18abd$, and $36ad$.
20. $35hkl$, $25ahkl$, and $65kl$.
21. $105abc$, $135bcd$, and $150bc$.

13. Addition and subtraction of numbers having a common factor.

How many times 5 is the sum of 3 times 5 and 4 times 5?

How many times a is the sum of 3 times a and 4 times a?

Or, combine $3a + 4a$ into a single number. Ans. $7a$.

How many times 5 is the remainder when 2 times 5 is subtracted from 6 times 5?

How many times a is the remainder when 2 times a is subtracted from 6 times a? Or, combine $6a - 2a$ into a single number. Ans. $4a$.

Rules. *The sum of numbers having a common factor is the sum of the coefficients of the common factor multiplied by the common factor.*

The difference between two numbers having a common factor is the difference of the coefficients of the common factor multiplied by the common factor.

For example, (1) $4a + 5a + 6a = 15a$.

(2) $2ab + 4ab + 7ab = 13ab$.

(3) $7a - 3a = 4a$.

(4) $10ab - 7ab = 3ab$.

Exercise 4

Combine the following:

1. $4a + 7a$.
2. $10a - 3a$.
3. $2ab + 3ab + 9ab$.
4. $12ab - 5ab$.
5. $7n + n + 3n$. Ans. $11n$.
6. $3a + 4a - 2a$. Ans. $5a$.
7. $2x + 5x - 6x$. Ans. x.
8. $5m - 3m + 7m$. Ans. $9m$.
9. $10ax + 12ax + 18ax$.
10. $2d + 5d - d - 3d$. Ans. $3d$.
11. $5m - 4m + 8m$.
12. $2x + 5x + 3x - 7x$.
13. $12ax + 7ax - 8ax$.
14. $2abc + 3abc - 4abc$.
15. $4xyz - 3xyz + 9xyz$.
16. $18mn - 4mn - 5mn$.

14. Exponents. If the product of $2a$ and $3b$ is $6ab$ (see § 7), then the product of $2a$ and $3a$ is $6aa$.

For convenience, $6aa$ is written $6a^2$, which is to be read "six a exponent 2". Similarly, aaa is written a^3, which is to be read "a exponent 3."

An **exponent** is a number written to the right and above a second number to indicate how many times the second number is to be taken as a factor.

$6a^2$ may also be read "six a square" and a^3 may be read "a to the third power" or "a cube."

The factors of $6a^2$ are 2, 3, a, and a.

Note. When the exponent is 1 it is omitted; $6a$ is the same as $6a^1$.

Exercise 5

Read and factor each of the following:

1. $8a^2$.
2. $12ab$.
3. $12x^2y$.
4. $15a^2b^2xz$.
5. $24a^3x^4z$.
6. $21a^3b^4c^2$.
7. $32x^4y^3w^2$.
8. $57w^3xyz^3$.
9. $6a^5b^4$.
10. $9c^8d^9$.
11. $5w^5x^4z^6$.
12. $18hl^3m^4n^2$.

Find the H. C. F. in each of the following:

13. $12a^2b$ and $16ab$.
14. $10a^2b$ and $15ab^2$.
15. $18a^2bc$ and $24ac^2$.
16. $20a^2bc$, $25ab^2c$, and $35abc^2$.
17. $15x^3y^2z$ and $35xy^2z^3$.
18. $18a^3b^2c$ and $48ab^2c^3$.
19. $24x^5y^3$ and $36x^2y^2$.
20. $42x^8y^5$ and $56x^5y^8$.
21. $x^{15}y^{10}$ and x^7y^{12}.
22. $x^8y^4z^6$ and $x^6y^8z^4$.

15. Division. Given one of the two factors of a number, **division** is the process of determining the other factor.

For example, "24 divided by 8 gives what quotient?" is the same as "8 times what number equals 24?" or, in symbols, "$8 \times ? = 24$."

"What is the quotient of $6ab$ divided by $2a$?" is the same as, "What number multiplied by $2a$ gives $6ab$?"

Therefore, in division separate, if possible, the dividend into two factors one of which is the divisor.

Exercise 6

Find the quotient in each of the following:

1. $12xy \div 4y$.
2. $18abc \div 3ac$.
3. $24abx \div 4ax$.
4. $20xyz \div 5xy$.
5. $30xy \div 10xy$.
6. $24abc \div 2ac$.
7. $18ax \div 9ax$.
8. $36a^2 \div 2a$.
9. $24a^2b \div 3ab$.
10. $32ab^2 \div 8ab$.
11. $27x^2yz \div 3xz$.
12. $30ab^2c \div 3abc$.
13. $40lm^2n \div 20mn$.
14. $28x^2y^2 \div 2xy$.
15. $24x^2y^2 \div 2x^2y$.
16. $32xy^2 \div 2xy$.
17. $30x^2y^2 \div 10xy^2$.
18. $12x^3y \div 6x^2y$.
19. $16x^3yz \div 2xyz$.
20. $36x^3y^7 \div 4x^2y$.

21. $\dfrac{15a^2b}{5ab}$. Ans. $3a$.

22. $\dfrac{8x^2y^2z^2}{2xyz}$

23. $\dfrac{25mnp}{5mp}$

24. $\dfrac{56a^2bc^2}{8ac}$.

25. $\dfrac{75m^2x^2}{25m^2}$.

26. $\dfrac{48abcd}{16acd}$.

27. Divide $8xy$ by $3ab$. Ans. $\dfrac{8xy}{3ab}$.

Note. If the divisor is not a factor of the dividend, the quotient may be written as a fraction, $\dfrac{8xy}{3ab}$, to be read "the fraction $8xy$ divided by $3ab$," or "$8xy$ over $3ab$."

28. Divide $10xy$ by $2mn$. Ans. $\dfrac{5xy}{mn}$.

29. Divide $12ab$ by $5cd$.

30. Divide $15x^2y$ by $3ay$.

16. Evaluation of literal numbers. In the formula $A = lw$ (see § 3), what is the value of A when l is 5 and w is 3? Or, what is the area of a rectangle whose length is five units and its width three units?

The substitution of a definite arithmetical value for each letter of a literal number is known as the **process of evaluation.**

Exercise 7

1. Given $a = 2$ and $b = 3$, what is the value of $a + b$? of ab? of $2a - b$? of $2ab$? of $4b - 5a$? of $3ab - 4$?

Given $a = 3$, $b = 4$, *and* $c = 2$, *find the value of each of the following expressions:*

2. $3a + 2b - 5c$.
3. $3abc - ab - ac$.
4. $2ab - ac + bc$.
5. $abc - a - b - c$.
6. $4abc - 3ac$.
7. $7ab + 2ac$.
8. If $n = 3$, what is the value of $3n + 4n$? of $7n$?
9. If $x = 3$, what is the value of $7x - 5x$? of $2x$?

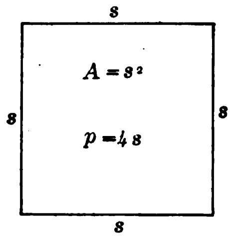
If $a = 3$ *and* $b = 4$, *find the value of each of the following in two ways:* (1) *by substituting in the problem as it is, and* (2) *by combining as in* § 13, *and then substituting:*

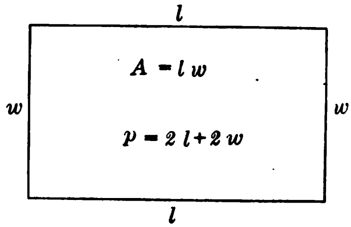
10. $3a + 4a + 2a$.
11. $7a + 3a - 4a$.
12. $2ab + 3ab - ab$.
13. $4b - 3b + 2b$.
14. $4ab - 2ab + 3ab$.
15. $6ab - 4ab + 2ab$.

Given $a = 2$, $b = 3$, $m = 5$, *and* $x = 1$, *evaluate each of the following:*

16. $2a^2b$.
17. $5abx$.
18. $8m^2x^2$.
19. $12a^2bm$.
20. $24a^3x^2$.
21. $30abmx^2$.
22. $5a^2b^3mx^2$.
23. $3a^2b \times 5mx$.
24. $12a^2b \div 3ab$.

17. Formulas. Many rules of arithmetic may be condensed into simple algebraic formulas. (For $A = lw$ and $d = tr$, see § 3.)



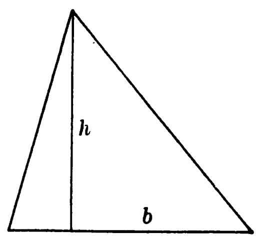


The formula for the area of a square is $A = s^2$.

The formula for the perimeter of a square is $p = 4s$.

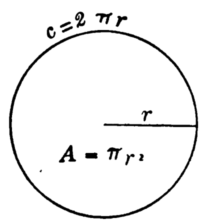
The formula for the perimeter of a rectangle is $p = 2l + 2w$.

The formula for the area of a triangle is $A = \frac{bh}{2}$.



The formula for the area of a circle is $A = \pi r^2$, where π is a number used in the formulas for circles and circular solids. Its approximate value is $\frac{22}{7}$, or 3.1416.

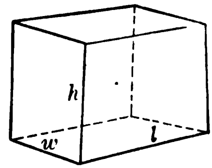
The formula for the circumference of a circle is $c = 2\pi r$.



The formula for the volume of a box is $v = lwh$.

The formula for the surface of a box is $s = 2lw + 2lh + 2wh$.

Ex. 1. What is the sum of the twelve edges of the box shown in the accompanying figure?



Exercise 8

Find the areas of the figures of the first eight exercises:

1. A rectangle whose base is 10 inches and altitude 14 inches.

2. A triangle whose base is 8 inches and altitude 14 inches.

3. A square one of whose sides is 9 inches.

4. A circle whose radius is 5 inches.

5. A circle whose radius is 10 feet.

6. A rectangle whose base is 7 inches and altitude a inches.

7. A circle whose radius is a inches.

8. A square whose side is a inches.

9. Find the number of cubic inches in the volume of a box that is one foot long, 8 inches wide, and 10 inches high.

10. Find the number of square inches of cloth required to line the sides and bottom of the box of No. 9.

11. How many square yards of plastering will be required for the walls and ceiling of a room that is 16 feet long, 12 feet wide, and 9 feet high, if an allowance of 9 square yards is made for doors and windows?

12. Which has the greater area, a circular window with a radius of 10 inches, or a rectangular window that is 20 inches long and 16 inches wide?

13. A goat is tied to a stake with a rope 11 feet long and can feed 1 foot beyond the end of the rope. Find the area of the space over which he may feed.

14. A rectangle and triangle have the same base, 14 inches. The altitude of the rectangle is 10 inches and that of the triangle is 20 inches. Find the area of each.

Exercise 9. Review

1. Write the sum of $2a$, $3b$, and $4c$.

2. Write the quotient in fractional form when the sum of a and b is divided by c.

3. Write the quotient when the product of a and b is divided by c.

4. John has a cents. If he earns b cents and spends c cents, how many cents has he left?

5. I have in my purse m quarters, n dimes, and 4 cents. How many cents have I in all?

6. The distance between two towns is $m + n$ miles. How many hours will be required for an automobile running 20 miles per hour to go from one town to the other?

7. A rectangle is x inches wide and y inches long. What is the perimeter?

8. What is the number of square inches in the area of the rectangle of No. 7?

9. An open box is x inches long, y inches wide, and z inches deep. How many cubic inches does it contain?

10. Find the number of square inches of cardboard necessary to make the box of No. 9, allowing 5 square inches for wastage.

11. Write the number that is 10 less than 7 times x.

12. If oranges cost d cents a dozen, what is the cost of one?

13. Write the result when the sum of a and b is divided by the sum of c and d.

14. What is the difference between a and b if a is the larger number?

15. If the sum of two numbers is 12 and one is 7, express the other using the minus sign.

16. If the sum of two numbers is 12 and one is x, what is the other?

17. If the sum of two numbers is a and one is x, what is the other?

18. If b eggs cost c cents, what is the price of one? of a dozen?

19. A boy works a hours at c cents per hour and spends his earnings for books that cost b cents each. How many books does he buy?

20. How many hours will be required to walk m miles at r miles per hour?

21. A man is y years old. How old was he 6 years ago? x years ago?

22. A man is x years old and his father is twice as old. Express the age of each 5 years from now.

23. James has 12 cents. If he spends 5 cents and earns 5 cents, how much will he then have?

24. Simplify $x - 6 + 6$; $12 - x + x$; $n - a + a$.

25. Factor completely $8a^3b^4$; $25m^2n^2x$; $49a^2b^3$.

26. Divide $63m^3n^3y^3$ by $7mny$.

27. If $a = 3$ and $b = 2$, find the value of $\frac{9b^2}{24a^3}$; of $\frac{20ab^2}{3a^2b}$.

28. If $x = 5$ and $y = 3$, find the value of $x^3 + y^3$ and of $x^3 - y^3$.

EQUATIONS

18. Definitions. The statement by the use of the symbol of equality, =, that two numbers are equal is called an **equation.** (See § 5.)

The = sign divides the equation into two parts called the **left,** or **first member,** and the **right,** or **second member.**

The student has already become familiar with such an expression as $n + 5 = 12$ and has found the value of n. He has also used several formulas, such as $A = lw$. (See §'s 3 and 17.)

An **equation of condition,** or a **conditional equation,** is an equation that is true for certain values only of the literal numbers.

$n + 3 = 7$ is true only when $n = 4$.

$2n - 3 = 7$ is true only when $n = 5$.

An **identity,** or an **identical equation,** is an equation that is true for all values of the literal numbers.

For example, $2a + 3a \equiv 5a$, which is to be read, "$2a + 3a$ is identically equal to $5a$."

Hereafter the **sign of identity** will be $\equiv$.

19. The **solution of an equation of condition** is the process of finding the **value,** or **values,** of the literal number, or numbers, that satisfy the equation. These values are called the roots.

The solution of an equation generally requires the use of one or more of the following mathematical truths, or **axioms:**

I. *If equals are added to equals, the sums are equal.*

The modified form of this axiom that is applied to the equation is:

The same number may be added to both members of the equation without destroying the equality.

That is, if $x - 4 = 7$, then $x - 4 + 4 = 7 + 4$, or $x = 11$.

II. *If equals are subtracted from equals, the remainders are equal.*

The modified form is:

The same number may be subtracted from both members of the equation without destroying the equality.

That is, if $x + 4 = 11$, then $x + 4 - 4 = 11 - 4$, or $x = 7$.

III. *If equals are multiplied by equals, the products are equal.*

The modified form is:

Each member of the equation may be multiplied by the same number without destroying the equality.

That is, if $\frac{1}{2}x = 4$, then $2 \cdot \frac{1}{2}x = 2 \cdot 4$, or $x = 8$.

IV. *If equals are divided by equals, the quotients are equal.*

The modified form is:

Each member of the equation may be divided by the same number (not zero) without destroying the equality.

That is, if $2x = 8$, then $2x \div 2 = 8 \div 2$, or $x = 4$.

Illustrative examples.

1. $3x - 2 = x + 6.$

$2x - 2 = 6.$ (Taking x from both sides. Axiom II.)

$2x - 2 + 2 = 8.$ (Adding 2 to both sides. Axiom I.)

Or, $2x = 8.$

$x = 4.$ (Axiom IV.)

The **root,** or answer, of an equation may be **checked,** or **proved,** by substituting the answer for the literal number in the original equation.

For example, if we substitute 4 for x in the above equation, we get $12 - 2 = 4 + 6$, or $10 = 10$. Since this **satisfies** the equation, we decide that 4 is a root.

Let the student attempt to check the equation with several other numbers, such as, $x = 2$, $x = 3$, and $x = 5$.

2. $5x + 2x - 10 = 8 + 3x - 6.$

$7x - 10 = 3x + 2$ (Combining. No axiom is used.)

$4x = 12.$ (What two axioms are used?).

$x = 3.$ (What axiom is used?).

Checking, $15 + 6 - 10 = 8 + 9 - 6$, or $11 = 11$.

Attempt to check with $x = 2$ and $x = 4$.

Exercise 10

Find the value of the literal number in each of the following equations and check. State the axioms used.

1. $6x - 2x + 3x = 49$.
2. $25 + 3n = 85 - n$.
3. $17 + x - 13 = 8 - x$.
4. $2x + 3 - x = 11 - x$.
5. $x + x + 1 = 39$.
6. $n + 2n + 3 = 45$.
7. $3y + 6 - y = 22 - 2y$.
8. $3x + 5 - x = 8 + x$.
9. $3n + 5n - 4n = 13 - 5$.
10. $3a + 2a - 5 = 13 - a$.
11. $4r + 3 = 2r + 4 + r$.
12. $5s - 5 = 25 - 5s$.
13. $8a + 7 = 7a + 8$.
14. $3n + 7 = 49 - 4n$.
15. $7x - 12 = 8 + 3x$.
16. $10n + 8 = 36 + 3n$.

20. Signs of aggregation. It is frequently necessary in algebra to use a number expression as a single number. This is indicated by the use of one of the **signs of aggregation.** These signs are the **parenthesis** ( ), the **bracket** [ ], the **brace** { }, and the **vinculum** $\overline{\quad}$. For example, $8(a - b)$ is to be read, "eight times the expression a minus b".

The signs of aggregation may be removed when the required operations are performed.

If a parenthesis containing two or more numbers connected by + or − signs is to be multiplied by a given number, it is necessary when removing the parenthesis to multiply each of the numbers within the parenthesis by the given number. That is, $6(5 + 4 - 2) = 30 + 24 - 12 = 42$.

This may be verified by first adding 5 and 4 and subtracting 2 and then multiplying this result by 6.

Simplify $3(a + b) + (a - b) + 5(2a + b)$.

Removing parentheses, $3a + 3b + a - b + 10a + 5b$.

Combining, $14a + 7b$.

Exercise 11

Remove parentheses and combine if possible:

1. $3(2+5)$.
2. $4(5-3)$.
3. $3(2a-b)$.
4. $4(a+b-c)$.
5. $5(3a-2b-c)$.
6. $7(2a-b-4)$.
7. $2(a+b)+3(2a+b)$.
8. $3(x+2y)+2(x-y)$.
9. $5(2n+3)+3(n-2)$.
10. $3(a+b+c)+2(a-b+c)+(a-b)$.
11. $2(x+y+2)+3(x+y-1)$.

Find the value of the literal number in each of the following equations and check:

12. $5x+3(2x-3)=18+2(x-2)-5$.

Solution. $5x+6x-9=18+2x-4-5$. **(Explain.)**
$11x-9=2x+9$.
$9x=18$. **(What axioms are used here?)**
$x=2$. **(What axiom is used?)**

Checking: $10+3(4-3)=18+2(2-2)-5$.
$10+12-9=18+4-4-5$.
$13=13$.

13. $13+2(2a-3)=15+3(a-1)$.
14. $6x+3(3-x)=27+2(x-6)$.
15. $12n+2(3n+1)=6(1+n)$.
16. $3r+7-r=8+r+2$.
17. $3(m+2)+2(m-3)=25$.
18. $2(s+1)+3(s+2)=28$.
19. $4x+3x-2x=25$.
20. $3(y-1)=2(y-1)$.
21. $5(2x-3)=3(3x-2)$.
22. $2(2x-3)=2x-3$.

Exercise 12. Problems

Study the following suggestions for problem solving:

(1) *Read the problem attentively.*

(2) *Find a literal number expression for each unknown.*

(3) *From the number expressions of* (2) *and the conditions of the problem, find two expressions for the same thing and use them as the members of an equation.*

(4) *Solve the equation and determine the unknowns.*

(5) *Check by showing that the results satisfy the conditions stated in the problem.*

1. The larger of two numbers is 2 more than 3 times the smaller and their sum is 26. Find the numbers.

Suggestion. If n represents the smaller number, what will represent the larger? The equation is $n + 3n + 2 = 26$.

2. Find two consecutive numbers such that the sum of 2 times the smaller added to 3 times the larger is 33.

Suggestion. Use x and $x + 1$ for the numbers. How do you represent 3 times $x + 1$?

3. The sum of two numbers is 12. Twice the first is 3 more than the second. Find the numbers.

Suggestion. If x represents the first number and their sum is 12, what will represent the second?

4. The sum of two numbers is 20 and the first is equal to 3 times the second. Find the numbers.

5. The number of boys in a certain algebra class is 7 less than twice the number of girls. Find the number of each if there are 23 pupils in the class.

6. A boy has $2.25 in dimes and quarters. If he has twice as many dimes as quarters, how many of each has he?

Suggestion. The equation is $10(2x) + 25x = 225$.

7. A grocer wishes to mix 40 pounds of coffee at a total cost of \$12.00. If he uses two grades of coffee, one costing 35 cents and the other 15 cents per pound, how many pounds of each must he use?

Suggestion. Let x and $40-x$ represent the numbers of pounds. The equation is $35x+15(40-x)=1200$.

8. A man made a will, leaving \$6,000 to be divided among three daughters and two sons. Each daughter was to receive \$500 more than each son. How much did each of the children receive?

9. Find two consecutive even numbers such that if the first be subtracted from twice the second, the remainder is 18.

10. Divide \$48.00 between B and C so that B may receive \$10.00 more than C.

11. Separate 47 into two parts such that one part is 2 more than 4 times the other part.

12. Charles, James, and Henry have 42 marbles. James has twice as many as Charles and Henry has as many as Charles and James together. How many has each?

13. The length of a rectangle is 3 times its width, and the perimeter is 20 inches. Find the length and width.

14. The length of a rectangle is 5 inches more than the width, and the perimeter is 50 inches. Find the dimensions.

15. The height of Washington Monument is 105 feet less than 3 times the height of Bunker Hill Monument. Find the height of each if the sum of their heights is 775 feet.

16. The weight of a cubic foot of iron is 10 pounds more than three times the weight of a cubic foot of limestone. The sum of the two weights is 610 pounds. Find the weight of each.

Exercise 13. Review

It is very important that the principles of this chapter be mastered thoroughly before the work of the next chapter is begun. The student should be able to do the following exercises with a fair degree of speed and accuracy.

1. The sum of two numbers is 15. If one is a, what is the other?

2. The difference between two numbers is d. If the smaller is x, what is the larger?

3. The difference between two numbers is d. If the larger is y, what is the smaller?

4. a is 3 more than b. Express the fact with an equation.

5. If a dimes are equal in value to b quarters, which is the larger number, a or b?

6. Does $x = 2.5$ satisfy the equation $2x + 3 = 4x - 2$?

7. Does $x = a$ satisfy $3(x + a) + 5(x - a) = 6a$? Does $x = a$ satisfy $2(x + a) + 3(x + 2a) = 13$?

8. If n is a whole number, show that $2n + 1$ is an odd number.

9. If n is a whole number, is $2n$ odd or even? Is $2n - 1$ odd or even?

10. If x yards of cloth cost 12 dollars, what is the cost of one yard?

11. James bought 20 pencils for m cents each and sold them for $m + 1$ cents each. How much did he gain?

12. A boy bought 100 pencils for c cents each and sold them for 5 cents each. If he gained, what was his gain?

13. Find the area of a rectangle whose length is 8 feet and width 5 feet. Use $A = lw$.

14. Find the area of a circle whose radius is 5 inches. Use $A=\pi r^2$.

15. Find the area of a triangle whose base is 10 feet and altitude 12 feet.

16. The distance a stone or other heavy article will fall in any number of seconds is expressed approximately by the formula $d=16t^2$, where d is the number of feet the object falls and t is the number of seconds. How far will a weight fall in 5 seconds? In a seconds?

17. A boy drops a stone from the top of a cliff overhanging a river into the water below. By the aid of a stop-watch he determines that it is 3.2 seconds from the time the stone leaves his hand till he sees its splash in the water. How high is the cliff above the water?

18. A merchant buys potatoes at b cents a bushel and sells them for p cents a peck. He gains g cents on one bushel. Express this fact by a formula.

19. Henry had $5x$ marbles and bought $2x-3$ more. How many did he then have?

Find the numerical values of the literal numbers and check:

20. $3n+2(2n+3)=5n+20.$

21. $7n+2(n-3)=7+2(n+4).$

22. $3(2x-3)=4(x-1)-1.$

23. $2(5-n)+4n=28.$

24. $3(2a+5)+7=2(a+19).$

25. I have in mind two numbers. One is $3(x+2)$ and the other is $2(x+3)$. If the first is 6 more than the second, find the value of x and evaluate the numbers.

26. If $7n-3$ is 16 more than $5n+1$, find the value of n.

27. Simplify $13 + 2(8 - 3) + 6(3 + 4)$.

28. Simplify $8x + 5y + 2(3x - 6) + 3(2y + 4)$.

If $a = 2$, $b = 3$, and $c = 4$, find the value of the following:

29. $3abc - ab - ac - a - b - c$.

30. $5a^2b^2 - 2ab^2$.

31. $2a^2 + 5ac + 3c^2$.

32. $a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$.

33. Write the product of $2a^2$, $3ab$, and $5c^2$.

34. What is the quotient when $24xyz$ is divided by $8xz$?

35. Divide $32a^2b^2c$ by $4abc$.

36. Factor $12bc$ into two factors. Can you do this in more than one way? What are the factors of $12bc$ when it is factored completely?

37. Factor completely $8abc$; $10rs$; $16xyz$; $24pk$.

38. What common factor do you find in $8abc$ and $12bck$?

39. Combine $3xy + 7xy - 4xy - 2xy + 8xy$.

40. The greater of two numbers is 5 times the less and their sum is 90. Find the numbers.

41. Divide 35 into two parts, such that the larger is 5 more than the smaller.

42. The sum of three numbers is 31. The second is 2 more than twice the first and the third is 1 less than three times the first. What are the numbers?

43. In a class containing 25 pupils the number of girls is 5 less than twice the number of boys. How many girls are there in the class?

44. A number plus twice itself plus three times itself equals 102. Find the number.

45. The difference between two numbers is 5 and their sum is 29. What are the numbers?

46. James had \$2.50 in dimes and quarters. If he has 3 more quarters than he has dimes, how many of each kind of coin has he?

47. Mary is 3 years older than Helen and Elizabeth is 5 years older than Mary. Find the age of each if the sum of their ages is 38 years.

48. A rectangle is 3 feet longer than wide. Find its dimensions if its perimeter is 26 feet.

49. The sum of three consecutive even numbers is 48. Find the numbers.

50. Three men contribute to the Red Cross. The second contributes twice as much as the first and the third gives \$5.00 more than the first and second together. Find the amount each gives if the sum of their contributions is \$155.00.

51. If to twice a certain number we add 11, the result is 17 more than the number itself. What is the number?

52. If to 3 times a certain number we add 10, the result is 100. What is the number?

53. Find three consecutive numbers such that the first plus twice the second plus 3 times the third equals 44.

CHAPTER II

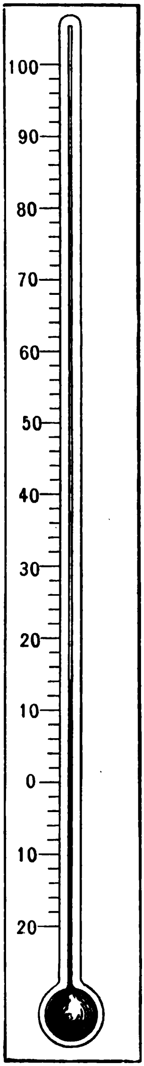
POSITIVE AND NEGATIVE NUMBERS

21. A reason for negative numbers. If a man owns property worth \$3,000 and has debts amounting to \$5,000, what will be the value of his property after his debts are paid? Or, what is his present worth?

Evidently the solution of such a problem in arithmetic is impossible since he does not have sufficient property to pay his debts; but in algebra we express the answer by the use of a new number, called a **negative number,** and say that the value of his property is $-\$2{,}000$ (read "negative \$2,000") which means that his present worth is \$2,000 less than nothing. In other words, he will have to earn \$2,000 in order to pay his debts.

22. The thermometer. That there is much to be gained by the use of $+$ and $-$ to indicate a property of numbers different from the operations of addition and subtraction is evident from the study of the thermometer.

The point at which the mercury stands in a thermometer under a certain agreed condition is marked as 0 (zero). The glass column above and below this point is marked off into equal divisions called degrees and these are numbered from the 0 point up and down the column. When the temperature is above 0, the reading is recorded as $+$ and when below, as $-$. Thus a temperature of $+20°$ means that the top of the mercury column stands just 20 degree spaces above 0 and one of $-20°$ means that



the mercury stands the same number of spaces below 0. Evidently $-20°$ does not name a quantity to be subtracted but locates a point on the column that is at the same distance from 0 as $+20°$ but in the opposite direction.

It is not customary to print the + sign before temperatures above zero. When no sign appears before a reading it is assumed to be above zero, or a + reading.

The following is the weather report for a day in January, 1920:

TEMPERATURE IN CHICAGO

[Last 24 hours.]

MAXIMUM, Noon 9
MINIMUM, 2 A.M................. —10

3 a. m.....	2	11 a. m...	8	7 p. m...	—4
4 a. m.....	2	12 Noon..	9	8 p. m...	—6
5 a. m.....	1	1 p. m...	8	9 p. m...	—7
6 a. m.....	1	2 p. m...	6	10 p. m...	—8
7 a. m.....	3	3 p. m...	4	11 p. m...	—9
8 a. m.....	5	4 p. m...	2	Midnight .	—9
9 a. m.....	7	5 p. m...	0	1 a. m...	—9
10 a. m.....	8	6 p. m...	—2	2 a. m...	—10

There is also another way of applying these two signs to the numbers of the thermometer. If we call motion up the column positive, or + motion, and motion down the column negative, or − motion, the movement of the mercury can be indicated by the use of + and −. The movement from 3 A. M. to 6 A. M. is downward 1° or $-1°$. From 6 A. M. to 9. A. M. it is upward 6°, or $+6°$.

Exercise 14

1. Express with positive or negative values the movement of the mercury from 6 A. M. to 10 A. M.

2. Express the movement from 3 A. M. to 3 P. M.

3. Express the movement from 9 A. M. to 9 P. M.

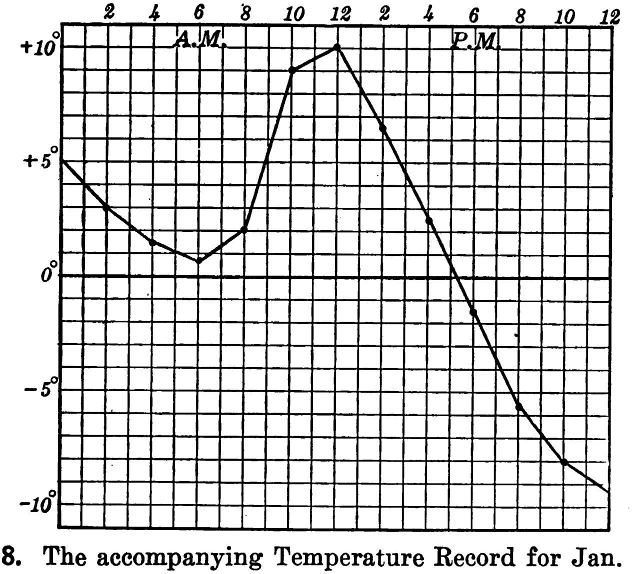
4. On the picture of a thermometer locate the points: $+12°, +3°, -5°, -8°$.

5. Locate the point $-3°$ and move $+8°$ from it.

6. Locate the point $+7°$ and move $-10°$ from it.

7. Begin at the zero point and move $+3°$, then $-8°$, then $-4°$, and then $+9°$.

Temperature Record, Jan. 8, 1918



8. The accompanying Temperature Record for Jan. 8th, 1918, indicates that the thermometer stood at $+3°$ at 2 A. M. and at $+1.4°$ at 4 A. M. Complete the readings

for each two-hour period until midnight. (Estimate the reading to nearest .1.)

9. What was the temperature at 9 A. M.? at 9 P. M.?

10. Read the Record for each hour from 8 A. M. to 4 P. M.

11. Read the Record for each three-hour period from midnight to midnight.

12. How many degrees did the thermometer rise or fall during the afternoon?

Explain how a + sign or a − sign may be used to indicate this movement.

13. What was the movement from midnight to 6 A. M.? from 6 A. M. to noon?

23. Other uses of + and −. East longitude is distinguished from west longitude by the use of + for east and − for west. Similarly, north latitude is distinguished from south latitude by the use of + for north and − for south. A ship in longitude $+12°$ is at the same distance east of the principal meridian that one in longitude $-12°$ is west. The first is 24° east of the second.

Ex. **1.** How many degrees of latitude apart are two ships, one in latitude $+10°$ and the other in latitude $-8°$?

Ex. **2.** How many degrees of longitude apart are two ships, one in longitude $+27°$ and the other in longitude $+44°$? If one is in longitude $-27°$ and the other is in longitude $-55°$?

Ex. **3.** What direction is the first from the second in each instance of Ex. 2 if in the same latitude?

Ex. **4.** The year since the birth of Christ is marked as + to distinguish it from time before Christ marked −.

How many years elapsed between -41 and $+70$?

Ex. **5.** Julius Caesar was born in the year -100. How long ago was that? He died in the year -44. How old was he when he died?

24. Definitions. Numbers which have the property of being opposite in direction or character are classified as **positive,** or **+ numbers,** and **negative,** or **− numbers.** Such numbers are sometimes called (1) **signed numbers,** (2) **directed numbers,** or (3) **+ and − numbers.**

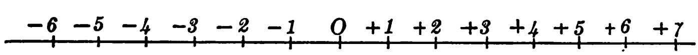
The **absolute** or **numerical value** of a positive or a negative number is the number of its units regardless of its character. The numerical value of both $+12$ and -12 is 12.

25. The double use of the signs + and −. We now have two uses for the signs + and −.

(1) *As signs of operation indicating addition and subtraction.*

(2) *As signs of character.*

For convenience, in distinguishing between these two uses as we determine the rules for adding, subtracting, multiplying, and dividing signed numbers, we shall make frequent reference to the following figure.



On the horizontal line, let the point O be taken as the zero point and, with a convenient unit, let equal distances be marked off to the right and left of O. Mark the points of division to the right $+1$, $+2$, $+3$, ..., and to the left -1, -2, -3, ... Let it be agreed that motion from left to right shall be positive or + motion, and from right to left negative or − motion.

The student should make a scale, following the plan of the preceding figure, that extends from -20 *to* $+20$.

If a moving point starts at 0 and goes 6 spaces to the right, or moves $+6$, it reaches the point $+6$. If it then moves $+2$, it reaches the point $+8$. If, starting at 0, it moves -5, then -2, then $+3$, it will be at the point -4.

26. Addition of signed numbers. By the use of the number scale, it is possible to combine, or add, $+$ and $-$ numbers. To combine such a group as $+6-3-2+5-3+8$, we start at 0, move 6 spaces to the right along the number scale, then 3 to the left, then 2 to the left, 5 to the right, 3 to the left, and 8 to the right. The resulting point, 11 spaces to the right, gives the answer $+11$.

Exercise 15

Combine by the number scale the following and read each answer as a positive or negative number:

1. $+2+6-5-4$.
2. $+4-6-3+2-5$.
3. $-3-2+4-3-1$.
4. $+8-3-4+2-5$.
5. $-4-6+3+6-1$.
6. $-2-7+3+5+1$.
7. $+3-5-2-7$.
8. $10+6-7-5-6+9$.
9. $-11+7+10-9$.
10. $-8-4+9+10$.

Note. The student will observe that in any one of these exercises the result may be obtained by making all the positive movements together as one movement, and then all the negative movements as one movement. This would reduce the first exercise to $+8-9$, and the movements would be $+8$, then -9, which gives the result -1.

Do each of the other exercises in a similar manner.

27. Rules. The preceding exercises illustrate the following:

(1) *To combine numbers having the same sign, find the sum of their numerical values and give it the common sign.*

(2) *To combine two numbers having opposite signs, find the difference of their numerical values and give it the sign of the one that is numerically the larger.*

(3) *To combine a series of positive and negative numbers, find the difference between the numerical sum of the positives and the numerical sum of the negatives and give it the sign of that sum that is numerically the larger.*

Exercise 16

Perform the following additions by the preceding rules without the use of the number scale:

1.	**2.**	**3.**	**4.**	**5.**	**6.**
$+8$	-4	$+9$	$+7$	$+6$	$+5$
-3	-2	$+1$	-6	-3	$+8$
-10	$+7$	-6	$+3$	-1	-11
$+8$	$+4$	-2	$+2$	$+11$	-7
-12	-3	-8	-10	-20	$+25$
-9. Ans.					

7.	**8.**	**9.**	**10.**	**11.**	**12.**
-20	$+8$	-7	-11	$+4$	-15
$+8$	-1	$+3$	-2	-14	$+7$
-4	$+9$	$+1$	$+6$	$+7$	-3
-10	-3	-8	-4	-6	$+2$
$+2$	$+9$	$+6$	-5	$+8$	-4
-7	-6	-9	$+22$	-5	-6

13.	**14.**	**15.**	**16.**
$+12ab$	$-3x^2y^3$	$-2(a-b)$	$+\$2.50$
$-10ab$	$+8x^2y^3$	$+4(a-b)$	-3.00
$-8ab$	$-4x^2y^3$	$-6(a-b)$	$+4.25$
$+12ab$	$-11x^2y^3$	$+3(a-b)$	-5.00
$-6ab$	$+3x^2y^3$	$-8(a-b)$	-2.25
0. Ans.			

28. Subtraction of + and − numbers. Express the amounts of the following problem in terms of + and − numbers and interpret the results:

At the beginning of the month a man's financial standing is as follows: on deposit in a bank \$95, in his purse \$27, due the grocer \$31, and due for rent \$35. He earns during the month \$240 and pays out for clothing \$55, for amusements \$12, and for other expenses \$100. What is his standing at the end of the month?

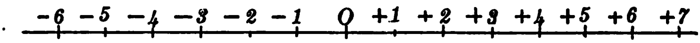
Discussion. If the \$27 in his purse were not included in the account, his money on hand at the end of the month would be decreased by \$27. That is, if a positive or + quantity is removed from the account, his present worth is decreased. In other words, the taking away of a + quantity is the same as the addition of a − quantity. In symbols this would be $-(+\$27)=+(-\$27)$.

We read $-(+\$27)$ "minus the expression positive \$27." Also $+(-\$27)$ is read "plus the expression negative \$27."

In like manner it appears that the removal, or subtraction, of a − number is the same as the addition of a + number. If the \$55 that he pays out for clothing were removed from the account (if no clothing were purchased), his present worth would be increased by \$55.

In symbols $-(-\$55)=+(+\$55)$. (How are these expressions to be read?)

Suppose we use the number scale to test the apparent conclusions of the preceding statements. If we wish to subtract -5 from $+7$, we ask how far in spaces it is on the scale from the point -5 to the point $+7$. We notice that it is necessary to move from -5 to 0, or a movement of $+5$, then to $+7$, or a movement of $+7$, making a total movement to the right of 12 spaces, or $+12$.



-5 subtracted from $+7$ gives the same result as $+5$ added to $+7$. In symbols, $+7-(-5)=+7+(+5)=12$. (Read carefully.)

Exercise 17

Make a number scale similar to the one in the preceding figure that extends to the right 15 or more spaces and to the left 15 or more spaces. By means of the scale you have made subtract the following:

1. -10 from $+7$.
2. $+10$ from 0.
3. -10 from -9.
4. $+10$ from $+9$.
5. -8 from -12.
6. -6 from -12.
7. -8 from $+9$.
8. -9 from $+8$.
9. -10 from 0.
10. -13 from -14.
11. -13 from $+14$.
12. -3 from $+11$.

Notice that the same result can be obtained in each of the preceding exercises by changing the sign of the subtrahend and combining it with the minuend as in algebraic addition.

In No. 1, -10 from $+7$ is the same as $+10$ added to $+7$.

In No. 2, $+10$ from 0 is the same as -10 added to 0.

In No. 3, -10 from -9 is the same as $+10$ added to -9.

Subtraction is the **reverse** of **addition.** It is the process of finding what number added to a given number called the **subtrahend** gives a second number called the **minuend.**

In No. 1, $+17$ added to -10 gives $+7$.

Similarly, what number added to $+6$ gives -9? Or, in other words, subtract $+6$ from -9. Notice that to

pass from $+6$ to -9 requires motion to the left, or negative motion, of 15 spaces. Written in symbols, $-9-(+6)$ is the same as $-9+(-6)$ or -15.

We now have two ways of stating the same problem:

$$(a)\ -9-(+6)=?$$
$$(b)\ +6+(?)=-9.$$

29. The preceding work makes clear the following:

Rule. *In subtraction, change the sign of the subtrahend and proceed as in addition.*

If the sign of a number is omitted, the $+$ sign is understood.

Exercise 18

Solve the following:

1. $7+(?)=10.$

2. $-3+(?)=5.$

3. $8+(?)=5.$

4. $-4+(?)=-6.$

5. $-5+(?)=-2.$

6. $7+(?)=-2.$

7. $5+(?)=-3.$

8. $-2+(?)=8.$

9. $-4+(?)=-10.$

10. $-8+(?)=-3.$

Write each of the preceding exercises in the (a) form and solve.

Perform the following subtractions:

11. $5-(+3)=?$

12. $6-(-3)=?$

13. $-4-(-7)=?$

14. $-7-(+5)=?$

15. $-8-(-2)=?$

16. $-9-(+7)=?$

17. $\begin{array}{r} 14 \\ \underline{7} \end{array}$ **18.** $\begin{array}{r} -14 \\ \underline{7} \end{array}$ **19.** $\begin{array}{r} 14 \\ \underline{-7} \end{array}$ **20.** $\begin{array}{r} -14 \\ \underline{-7} \end{array}$ **21.** $\begin{array}{r} 5 \\ \underline{8} \end{array}$

22. $\begin{array}{r} 5 \\ \underline{-8} \end{array}$ **23.** $\begin{array}{r} -5 \\ \underline{-8} \end{array}$ **24.** $\begin{array}{r} -5 \\ \underline{8} \end{array}$ **25.** $\begin{array}{r} -3a \\ \underline{8a} \end{array}$

26. $\begin{array}{r} -8n \\ -3n \\ \hline \end{array}$ **27.** $\begin{array}{r} -11n \\ -17n \\ \hline \end{array}$ **28.** $\begin{array}{r} 4abc \\ -8abc \\ \hline \end{array}$ **29.** $\begin{array}{r} 18xyz \\ -20xyz \\ \hline \end{array}$

Combine each of the following:

30. $12ab + (-3ab)$. Ans. $9ab$.

31. $12ab - (3ab)$.

32. $12ab - (-3ab)$.

33. $7a + (-2a) - 6a$.

34. $-7bc + (9bc)$.

35. $-7bc - (-9bc)$.

36. $-7bc + (-9bc)$.

37. $8 - (-8) - 8$.

38. $3 + 4 - 6 - 7$.

39. $\begin{array}{r} -4n \\ 5n \\ -2n \\ \hline \end{array}$ **40.** $\begin{array}{r} 3x \\ -2x \\ -4x \\ \hline \end{array}$

41. $\begin{array}{r} 2abc \\ -\ abc \\ 3abc \\ -4abc \\ \hline \end{array}$

42. $\begin{array}{r} -7a \\ 2a \\ 3a \\ -4a \\ 9a \\ \hline \end{array}$

43. $\begin{array}{r} -12xy \\ 7xy \\ 12xy \\ -\ 3xy \\ -\ 6xy \\ \hline \end{array}$

44. $\begin{array}{r} 8 \\ -3 \\ -9 \\ 7 \\ -2 \\ 9 \\ \hline \end{array}$

45. $\begin{array}{r} 7 \\ -5 \\ -6 \\ 8 \\ -9 \\ 7 \\ -3 \\ 4 \\ -5 \\ \hline \end{array}$

Exercise 19. Problems

1. If to his present worth at the beginning of a month, a man adds his salary of \$300, pays living expenses of \$130, and then has left a present worth of \$120, what was he worth at the beginning of the month?

Solution. Let x = the number of dollars in his present worth at the first of the month.

Then $x + 300 - 130 = 120$, according to the conditions of the problem.

Solving for x gives $x = -50$, or the number of dollars that he was worth at the first of the month.

He was, therefore, \$50 **in debt** at the beginning of the month.

2. A steamer moves 10 miles per hour upstream against a current whose rate is 2.5 miles per hour. What is the rate of the steamer in still water?

How far upstream would the steamer go in one hour if the rate of the current were 12.5 miles per hour? 15 miles per hour?

3. On a certain day in January the thermometer rises 2 degrees per hour for each hour after 7 A. M. until 1 P. M. It remains stationary until 4 P. M., then falls 3 degrees per hour until midnight. It then reads $-6°$. What was it at 7 A. M.? At 7 P. M.?

4. An aeroplane whose speed in quiet weather is 90 miles per hour, can make 30 miles in 2 hours against a hurricane. Find the velocity of the wind.

5. On another day the same machine makes 360 miles in 3 hours, flying with the wind. Find the velocity of the wind.

6. The thermometer registers 3 degrees higher on Tuesday than on Monday, 3 degrees higher on Wednesday than on Tuesday, and so on for the remaining days of the week. If it registers 75° Saturday, what was the temperature Monday? Wednesday?

7. Tuesday the temperature was 12° higher than it was Monday. Wednesday it was 15° higher than Monday. If the sum of the three temperatures was 30°, find Monday's temperature.

8. At the beginning of the month A had 4 times as much money in the bank as he had in his pocket. He paid several bills amounting to $90 and had left altogether $85. How much money did he have in his pocket at the beginning of the month?

9. A man, who can row a boat 6 miles per hour in still water, pulls against a current of 8 miles per hour. Where will he be in 3 hours? If he rows with the current, where will he be in 5 hours?

10. A's money exceeds B's by \$25 and B has \$40 less than C. All three together have \$185. How much has each?

11. A sailing ship tacks seven times during its trip and its changes in longitude for the changes in direction were, $-2°$, $1°$, $-3°$, $2°$, $-1°$, $4°$, and $-5°$. If its longitude at the beginning of its voyage was $-27°$, what was it at the end?

12. Find the latitude of the ship of Problem 11, if its changes in latitude were $3°$, $2°$, $-1°$, $2°$, $-3°$, $-2°$, and $-4°$, its initial latitude being $+12°$.

13. A man travels eastward along a road that goes through a town called B. If he travels three hours at the rate of 10 miles per hour and finds that he is 18 miles east of B, how far was he east of B at the beginning of his journey?

Note. Solve, using the equation $x+30=18$, and interpret the negative result.

14. The sum of three numbers is 9. The second is 5 less than 2 times the first, and the third is 2 more than 3 times the first. Find the numbers.

15. B's present worth is \$25 less than A's, and C's present worth is equal to the sum of A's and B's. All three together have the equivalent of \$10. Find the present worth of each.

16. Wednesday's average temperature was 13° colder than that of Tuesday and the average temperature of Thursday was 10° warmer than that of Wednesday. Find the three temperatures if their sum was 8°.

30. Multiplication of + and − numbers. We learned in arithmetic that multiplication is a short process of adding numbers that are all alike.

$(+2) \times (+5)$ means that two 5's are to be added, or that we are to perform the operation of $5 + 5$. Therefore $(+2) \times (+5) = +10$.

Similarly $(+3) \times (+7)$ means $+7 + 7 + 7$, which equals 21.

Rule. *The product of two positive numbers is a positive number.*

In the same manner $(+3) \times (-7)$ is the same as $(-7) + (-7) + (-7)$, which equals -21.

Rule. *The product of a positive number and a negative number is a negative number.*

Since $(-7) \times (+3)$ is the same as $(+3) \times (-7)$, which equals -21, we have the,—

Rule. *The product of a negative number and a positive number is a negative number.*

$(-2) \times (-5)$ means that -5 is to be subtracted two times. If we do this on the number scale, starting at zero, we find that the result is $+10$.

Rule. *The product of two negative numbers is a positive number.*

The rules for signs in multiplication may now be written as one.

Rule. *The product of two numbers having like signs is positive and the product of two numbers having unlike signs is negative.*

Note. The product of three or more factors may be found by getting the product of two of the factors and then multiplying this result by the third, etc.

Exercise 20

Multiply the following:

$$\textbf{1.}\ \begin{array}{r} 7 \\ \underline{-7} \end{array} \qquad \textbf{2.}\ \begin{array}{r} -7 \\ \underline{7} \end{array} \qquad \textbf{3.}\ \begin{array}{r} -7 \\ \underline{-7} \end{array} \qquad \textbf{4.}\ \begin{array}{r} a \\ \underline{a} \end{array} \qquad \textbf{5.}\ \begin{array}{r} a \\ \underline{-a} \end{array} \qquad \textbf{6.}\ \begin{array}{r} -a \\ \underline{a} \end{array}$$

$$\textbf{7.}\ \begin{array}{r} -a \\ \underline{-a} \end{array} \qquad \textbf{8.}\ \begin{array}{r} 2a \\ \underline{2a} \end{array} \qquad \textbf{9.}\ \begin{array}{r} 2a \\ \underline{-2a} \end{array} \qquad \textbf{10.}\ \begin{array}{r} -2a \\ \underline{2a} \end{array} \qquad \textbf{11.}\ \begin{array}{r} -2a \\ \underline{-2a} \end{array}$$

$$\textbf{12.}\ \begin{array}{r} 3x \\ \underline{-5x} \end{array} \qquad \textbf{13.}\ \begin{array}{r} 2a \\ \underline{-3b} \end{array} \qquad \textbf{14.}\ \begin{array}{r} 3ab \\ \underline{-2ac} \end{array} \qquad \textbf{15.}\ \begin{array}{r} 4ab \\ \underline{-4a} \end{array} \qquad \textbf{16.}\ \begin{array}{r} -7ab \\ \underline{-7ax} \end{array}$$

$$\textbf{17.}\ \begin{array}{r} -6bcd \\ \underline{-6bcx} \end{array} \qquad \textbf{18.}\ \begin{array}{r} -2xyz \\ \underline{2xyz} \end{array} \qquad \textbf{19.}\ \begin{array}{r} -3abc \\ \underline{6abd} \end{array} \qquad \textbf{20.}\ \begin{array}{r} 2xy \\ \underline{3ac} \end{array}$$

Find the answer for each of the following:

21. $(3)\cdot(-2)\cdot(4) = ?$

22. $(2)\cdot(-6)\cdot(-2)\cdot(7) = ?$

23. $(2x)\cdot(-3x)\cdot(z) = ?$

24. $(3)\cdot(2)\cdot(-7)\cdot(-2x) = ?$

25. $(2a)\cdot(3b)\cdot(-6a) = ?$

26. $(2ab)\cdot(3ab)\cdot(-4ab) = ?$

31. Division of positive and negative numbers. Since division is the process of finding what number times the divisor gives the dividend, the rules are the same as in multiplication.

For example, if 6 times -7 equals -42, then -42 divided by 6 equals -7, or -42 divided by -7 equals 6.

Rule. *If the dividend and divisor have like signs, the sign of the quotient is positive. If they have unlike signs, the sign of the quotient is negative.*

Exercise 21

Perform the following divisions:

1. $\frac{12}{3}$. 2. $\frac{12}{-3}$. 3. $\frac{-12}{3}$. 4. $\frac{-12}{-3}$. 5. $\frac{6a}{2}$. 6. $\frac{6a}{-2}$.

7. $\frac{-6a}{3a}$. 8. $\frac{-6a}{-3a}$. 9. $\frac{24abc}{-6ab}$. 10. $\frac{12ab}{-3a}$. 11. $\frac{9axy}{3xy}$.

12. $\frac{24xyz}{-6x}$. 13. $\frac{-15axy}{-3y}$. 14. $\frac{18a^2}{-6a}$. 15. $\frac{20x^2y^2}{-5xy}$.

16. $\frac{14a^2b^2c^2}{-7ac}$. 17. $\frac{-20a^3b}{-4a^2}$. 18. $\frac{20a^2b^2}{2a^2b}$. 19. $\frac{48mnp}{-6p}$.

20. $\frac{-18abcd}{-6cd}$. 21. $\frac{72ab^2x^3z^2}{9abxz}$. 22. $\frac{24xy}{5xy}$. Ans. $\frac{24}{5}$.

In the last exercise the numerator cannot be divided by the denominator without a remainder, or fractional result. When this occurs it is necessary to follow the method for reducing, or simplifying a fraction, which we learned in arithmetic, by dividing all the common factors out of both numerator and denominator.

In No. 22, x and y are common factors.

Similarly, $\frac{-12abcd}{-9abx}=\frac{4cd}{3x}$. Notice that -3, a, and b are all of the common factors.

23. $\frac{13xyz}{5xy}$. 24. $\frac{-25abcx}{-15axz}$. 25. $\frac{-30a^2b^2c^3}{-18abcd}$.

26. $\frac{28x^2y^4z^2}{21axy}$ 27. $\frac{-33m^3n^3p^3}{-11bn}$. 28. $\frac{54k^4s^3t^2}{24ks^2t}$.

32. Equations. Nearly all of the equations which we have solved so far have had positive roots. But the solution of an equation frequently gives a negative root, and, since the negative root may have a meaning under the conditions of the problem (see § 21), we are now ready to solve for both positive and negative roots.

Consider the following four equations with their roots.

I. $3x=6$,	II. $3x=-6$,	III. $-3x=6$,	IV. $-3x=-6$,
$x=2$.	$x=-2$.	$x=-2$.	$x=2$.

Notice that to obtain the root in I we divided each member by 3, in II by 3, in III by -3, and in IV by -3.

Exercise 22

Solve the following equations and check:

1. $4x+3(x-2)=9x+18$.

Solution. $4x+3x-6=9x+18$. How obtained?
$-2x=24$. How obtained?
$x=-12$.

Check. $4(-12)+3(-12-2)=9(-12)+18$.
$-90=-90$.

2. $2(x-4)+3(x-3)=13$.
3. $7x+13=9x+17$.
4. $3(x+2)-8=5(x-3)+17$.
5. $3x-5=5x-3$.
6. $3(x+2)+2(x+3)=-x$.
7. $5(z+2)=2(z-2)-4$.
8. $2(2x-3)+5=5(x-1)+2$.
9. $12-3x=24+3x$.
10. $5(a+2)+2(5-a)=70+5a$.
11. $33+3(5-n)=18-6n$.

12. $4(2n+3)+2(2n-4)=5(2n+1)$.

13. $2(2x-3)=3(3x-2)$.

14. $2(3a-1)+3(6a+1)=9a+6$.

15. $4(2r+1)+8r=2(2r-1)$.

16. $3(2n-5)=10-4n$.

17. $6(5-2n)+25+34n=0$.

18. $3(m-1)+6(2m+3)=3m+11$.

19. $2x+8-3x=13+5(x-1)$.

20. $5x+8=8x+3(2-2x)$.

Exercise 23. Review

1. Find the product of a, $-3b$, and ab; of $2a$, $-3a$, and $-bc$.

2. Find the product of $2a$, $-3b$, ax, and $-7bx$. Ans. $42a^2b^2x^2$.

3. Write ten pairs of factors of 48, such as 1 and 48, -1 and -48, 2 and 24, etc.

4. How many pairs of factors of 144 can you write?

5. Write as many sets of three factors each as possible of 24. (4, 2, and 3; -4, 2 and -3; -1, 8, and -3; etc.)

6. Write six pairs of factors of $42a^2b^2x^2$.

7. Find the product of abx, $-2abx$, and $3abx$.

8. Write four sets of three factors each of $-6a^3b^3x^3$.

9. Find the product of $4ax$, $-3ab$, $-2cd$, and $-5bc$. Separate this product into two factors; three factors; five factors.

Factor it completely. How many factors in each set may be negative?

If $a=2$, $b=-3$, $x=1$, and $y=4$, evaluate the following:

10. $\frac{ab}{x}$ **11.** $\frac{ab^2}{y}$. **12.** $\frac{abx^2}{xy}$. **13.** $\frac{4a^3bx}{3xy}$. **14.** $\frac{-2a^3b^2y}{2y}$

15. Find the value of $(-2)^2$, $(-2)^3$, $(-2)^4$, and $(-2)^5$.

16. Add by using the number scale: (See § 26.) $-3, 7, 2, -5, 3$, and -4. Also $-12, 8, -3, 5, 4$, and -2.

17. A man saves \$300 in three months. If he saves \$150 more the second month than the first and \$150 more the third than the second, what was the amount saved each month?

18. New Orleans is 65° north of Buenos Aires and St. Paul is 15° north of New Orleans. If the sum of the latitudes of the three cities is 40°, find the latitude of each.

19. St. Louis is 15° west of New York and San Francisco is 32° west of St. Louis. The sum of the longitudes of the three cities is $-287°$. What is the longitude of each?

20. A water tank whose capacity is 20,000 gallons has one inflow pipe discharging 2,000 gallons per hour, and two outflow pipes discharging 1,200 and 1,800 gallons, respectively, per hour. If all three pipes are opened and the tank is empty in what time will it be filled? If the tank were full and all three discharging, what would happen?

21. If the inflow pipe in No. 20 were closed and the tank were full, in what time would the tank be emptied if both outflow pipes were opened? the smaller alone? the larger alone?

22. The sum of three consecutive odd numbers is 39. What are they?

23. A man on a walking trip is able to make 3 miles more the second day than the first, 4 less the third than the second, and 5 miles more the fourth than the third. If he walks 64 miles in the four days, find his distance for each day.

24. One of two numbers is 5 times as large as another. If 10 is subtracted from the larger and added to the smaller, the results are equal. What are the numbers?

25. Divide $100 between two men so that one shall receive $10 more than one-half what the other receives.

Note. Let $2x$ and $x + 10$ represent the respective numbers of dollars.

26. 18 is one-half the sum of what three consecutive even numbers?

27. The sum of three numbers is 4. The second is 7 more than the first and the third is 5 more than twice the first. Find the numbers. Answers: -2, 5, and 1.

28. John is twice as old as James. If 3 years ago he was three times as old, what are their present ages?

Suggestion. Use $2x$ and x for their present ages. Then $2x - 3$ and $x - 3$ would be their ages three years ago. What is the equation?

29. A father is twice as old as his son and 10 years ago he was three times as old. What are their ages?

30. A father is 50 years old and his son is 20 years old. How long since the father was three times as old as his son? Four times as old?

31. Find the average of the numbers 6, 8, -3, and 5.

Note. The **average** of several numbers is the quotient of their sum divided by the number of numbers.

32. Find the average of the numbers 7, 3, -8, and 10; of $3a$, $-5a$, and $11a$.

33. The temperatures recorded at 7 A. M. daily for five consecutive mornings in January, 1918, were 12°, $-1°$, $-17°$, $-10°$, and 2°. What was the average temperature for the five mornings?

34. Solve for x: $3(x+7)=2(1-3x)-8$.

35. Solve for m: $5(3-m)+7=7m+8(m-4)-1$.

36. Solve for x: $3(x+a)+2(2x-a)=15a$. Ans. $x=2a$.

Suggestion: Collect terms as though a were a known number.

37. Solve for y: $3(y-n)+2(2n-y)=5n$.

38. Solve for n: $5(2n-a)+3(3n+2a)=13n+13a$.

39. Solve for R: $2(3R-4)+2(2R-1)=8R-4$.

40. Solve for x: $3(x-a)+5(2x-a)=5a$.

Perform the indicated operations:

41. $a \cdot a \cdot (-b) \cdot (-c) \cdot (-2) \cdot b^2 \cdot (-4) = ?$

42. $(-xy) \cdot (-yz) \cdot (-xz) \cdot (-x) \cdot (-y) = ?$

43. $2abc \cdot (-2ab) \cdot (-2ac) \cdot (-2bc) = ?$

44. $(-12xyz) \div (2xy) = ?$

45. $(-12xyz) \div (-2xy) = ?$

46. $\frac{-64xyz}{8xz}$. **47.** $\frac{-3abc}{48a^2bc^3}$. **48.** $\frac{-100abcd}{-48bdxy}$. **49.** $\frac{-729a^3}{-144ab}$.

50. When is the sum of several like terms positive? negative?

When is the difference of two like terms positive? When negative?

When is the product of several terms positive? When negative?

When is a quotient positive? When negative?

CHAPTER III.

FUNDAMENTAL PROCESSES

33. Definitions. A **term** is a number, or an algebraic expression, that does not consist of two or more parts connected by the signs $+$ or $-$.

5, $2a$, $8b^2$, and $9a^2x$ are terms.

A **monomial** is an expression of one term.

A **polynomial** is an expression of two or more terms connected by the signs $+$ and $-$.

An expression of two terms is also called a **binomial** and one of three terms is called a **trinomial.** Thus $a - b$ may be read "the binomial $a - b$", and $x - y + z$ may be read "the trinomial $x - y + z$."

Like or similar terms are terms that have the same literal factors with the same exponents. They may have different numerical coefficients. $2ab$, $7ab$, $9ab$, and ab are similar terms. Likewise $2a^2x$, $5a^2x$, $125a^2x$, and a^2x are similar terms.

An expression enclosed in a symbol of aggregation is considered as a single term until the operation of removing the sign of aggregation is performed. $(a - b)$ and $(2a - 3b + 4c)$ are such terms. $8a(a - 2b)$ is also a single term, to be read "$8a$ times the expression $a - 2b$". $3b(x - y + 3)$ is a single term, to be read "$3b$ times the expression $x - y + 3$".

ADDITION AND SUBTRACTION

34. Addition of polynomials. The student has already learned that he may find the sum of two or more polynomials by enclosing them in parentheses preceded by the $+$ sign. Thus the sum of $3a + 5b$ and $8a + 3b$ is found as follows:

$$(3a + 5b) + (8a + 3b) = 3a + 5b + 8a + 3b = 11a + 8b.$$

That there is a shorter and more convenient method for finding the sum of two or more polynomials is evident from a study of the method for adding denominate numbers:

$$\begin{array}{cc} \text{pounds} & \text{ounces} \\ 3 & 5 \\ 8 & 3 \\ \hline 11 & 8. \end{array} \text{ Ans.} \qquad \text{Similarly,} \quad \begin{array}{r} 3a + 5b \\ 8a + 3b \\ \hline 11a + 8b. \end{array} \text{ Ans.}$$

$$\begin{array}{r} 3a - 2b \\ -2a + 3b \\ \hline a + b. \end{array} \text{ Ans.} \qquad \text{Or the sum or} \quad \begin{array}{rrr} a & + 2x & - c \\ 2a & - 4x & + 3c \\ -4a & - 5x & + c \\ \hline -a & - 7x & + 3c. \end{array} \text{ Ans.}$$

Rule. *Place similar terms in the same column.*

Find the algebraic sum of the terms in each column and write these results, each with its proper sign, in a new polynomial.

Exercise 24

Add the following:

1. $\begin{array}{r} 4a - b \\ -a + b \\ \hline 3a. \end{array}$ Ans. **2.** $\begin{array}{r} 2x - 3y \\ -x + 4y \\ \hline \end{array}$ **3.** $\begin{array}{r} 4m + 5n \\ -3m - n \\ m - 3n \\ \hline \end{array}$ **4.** $\begin{array}{r} -3s + 2t \\ 4s - 3t \\ 2s + t \\ \hline \end{array}$

5. $\begin{array}{r} 4m + 2n \\ -2m + n \\ -m - 3n \\ \hline \end{array}$ **6.** $\begin{array}{r} 3x - 4z + 7u \\ -2x + z + 2u \\ -5x + 3z - 5u \\ \hline \end{array}$ **7.** $\begin{array}{r} 2ab - 5cd + 3ax \\ -ab + 2cd + 2ax \\ 3ab + 4cd - 7ax \\ \hline \end{array}$

Find the sum in each of the following by arranging in columns and adding:

8. $ax + 2xy - 3xz$, $ax + 2bx - 3xy + xz$, and $4ax + 2bx - 3xz$.

9. $2ac + 3ab - 3xz + cd - a$, $4ab - 4cd + 4a$, $2ac - 3xz + 3cd$, $4a + 3ac - ab + 6xz$, and $ac - ab + xz + ca - a$.

10. $2a^2 + 3a - 4$, $a^2 - 3a + 2$, $-3a^2 + 5a - 2$, and $a^2 + 5$.

11. $x^2 - xy + y^2$, $x^2 - 2xy + y^2$, $x^2 + 2xy + y^2$, and $x^2 - y^2$.

12. $a^3 - 3a^2 + 3a - 1$, $a^3 + 3a^2 - 5$, $3a^3 - 2a^2 - a + 3$, $a^3 - 4a + 4$, $-7a^3 - 3a^2 + 9a + 10$, and $a^3 + 12$.

13. $-12a + 15b - 3c$, $-12b - 5c + 3a$, and $-12c - 10a + 5b$.

14. $2a + 3b + 4c$, $2b + 3c + 4a$, and $2c + 3a + 4b$.

15. $8 + 6x^3 - 4x^2 - 3x$, $3x - 5x^3 + 3x^2 - 2$, and $x^2 - x + 2$.

16. $x^3 - y^3$, $x^3 - 2xy^2$, $6x^2y + y^3$, $x^3 + xy^2 - x^2y + y^3$, $x^2y - xy^2$, $-3x^2y + 3xy^2 - x^3 + y^3$, and $-xy^2 + 3x^2y$.

17. $A^3 - 3AB^2 + 3A^2B - B^3$, $A^3 - B^3$, $2A^2B + 3AB^2$, $B^3 + 2A^2B + 4AB^2 - 3B^3$, and $-4A^3 - 4A^2B + 4AB^2 + B^3$.

18.
$$\begin{array}{r} 2(a+b) - 3(c-d) \\ -(a+b) + 2(c-d) \\ (a+b) + (c-d) \\ \hline \end{array}$$

19.
$$\begin{array}{r} 2(m-n) - 3(x+y) \\ (m-n) + 2(x+y) \\ -3(m-n) + 2(x+y) \\ \hline \end{array}$$

20. $2(a-b) - 3(c+d)$, $-4(a-b) + 5(c+d)$, and $3(a-b) + (c+d)$.

35. Subtraction of polynomials.

Rule. *Place similar terms in columns, the subtrahend under the minuend.*

Change the sign of each term of the subtrahend and combine as in addition. (See § 29.)

Note. If the change of signs can be made mentally, it is much better. But, if necessary to write the new signs, place them underneath the old.

Exercise 25

Subtract the following:

1.
$$\begin{array}{rrr} 2a & -3b & -7c \\ -3a & +2b & -\ \ c \\ + & - & + \\ \hline 5a & -5b & -6c. \end{array} \quad \text{Ans.}$$

2.
$$\begin{array}{rrr} -4x & +3y & -\ \ z \\ -2x & -7y & +6z \\ \hline \end{array}$$

3.
$$\begin{array}{rrrr} 2ab & +3ac & -bc & -5 \\ ab & -\ \ ac & & +3 \\ \hline \end{array}$$

4.
$$\begin{array}{rrr} 2xy & +3xz & -8 \\ 6xy & -6xz & +9 \\ \hline \end{array}$$

5. Subtract $2ab - 3ac - 7bc + 2$ from $4ab + 4ac - 13$.

6. From $x + 2y + 3z$ take $2x + 3y + 4z$.

7. $x^3 - 3x^2y + 3xy^2 - y^3$ taken from $x^3 - y^3$ leaves what?

8. Subtract $a^2 - 2ab + b^2$ from $a^2 + 2ab + b^2$.

9. Subtract $a^3 - 3a^2b + 3ab^2 - b^3$ from $a^3 + 3a^2b + 3ab^2 + b^3$.

10. From the sum of $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ take $2a^2 - 2b^2$.

11. From the sum of $2ax - 3xz$, $-3xz + 2xy$, and $ax + xz + xy$ take the sum of $-2ax + 2xz$, $xy - 2xz + ax$, and $ax - xz - xy$.

12. From $3(a - b) + (c + d)$ take $2(a - b) - (c + d)$.

Given $M = a^2 - ab + b^2$, $N = a^2 + ab + b^2$, $R = a^2 + b^2$, *and* $S = a^2 - b^2$, *find the following:*

13. $M + N - R$. **14.** $M - R - S$. **15.** $2M - N + R$.
16. $M + N - R - S$. **17.** $2(M + N) - R - S$.

36. Subtraction indicated by the use of the signs of aggregation. A $-$ sign before any one of the signs of aggregation means that the expression within the sign of aggregation is to be subtracted.

$3ab + 2ac - (6ab + ac)$ is the same exercise as, "from $3ab + 2ac$ subtract $6ab + ac$." Or $3ab + 2ac - (6ab + ac) = 3ab + 2ac - 6ab - ac = -3ab + ac$.

If there is a coefficient before the sign of aggregation, multiply each term within the sign by this coefficient.

Note. See § 20 for parentheses preceded by the $+$ sign.

Rule. *To remove an expression from a sign of aggregation preceded by the minus sign, multiply each term by the coefficient (if there is one), and change the sign of each resulting term.*

Exercise 26

Remove the parentheses and combine terms:

1. $2(3 + 4) - (5 - 3) - 7$. Ans. 5.
2. $A - 2(A + 2B) + 3(B - 2A)$.
3. $5(2 - 3) - (5 - 3) + 2(-2 + 3)$.
4. $13 - 3(-3 + 2) - 2(5 - 2)$.
5. $5(2x - 3) - 3(3x - 2) - (5 - x)$.
6. $5x - (2x + 3y) + 2y$.
7. $18 - 2(3 - 2) + 3(-2 + 5)$.
8. $21 - 4(-2 + 3) + 2(-1 - 7) + 36$.
9. $3n - 2(2n - 1) + 5(3n - 4)$.

10. $-(a+b)-2(2a-b)+8a$.

11. $-5(x-y-z)+3(2y-z+3x)$.

12. $7(m-n)-5(2m+n)-3(-m-n)$.

13. $4(x-3)-5(y-3)+6(z-3)$.

14. $-n(m-n)+m(2n-3m)+9mn$.

15. $-2a(a+3b)+5b(b-c)-4c(a-b)$.

Solve the following equations and check:

16. $2(x+5)-4=11+x$.

17. $3(n+2)+2(n-3)-13=17$.

18. $5(x-2)-2(2x+7)=8-x$.

19. $5(n+8)-7n=64+n$.

20. $2(x-3)-5(2x+1)=4-11x$.

21. $2(2a-4)-3(a+2)=7(6-a)$.

22. $x(15-13)-2(2x+1)=17x+93$.

23. $-2(2n+3)-6(8-n)=3(2n+4)-14$.

24. $2x+3(x-5)-2(2x-3)=-4$.

25. $5(a+4)-3(2+a)-4(a-1)-8=0$.

26. $7x-5(3-2x)=72-2(3+5x)$.

27. $13+3n=12-4(1-n)$.

Given $K=x^2+x+1$, $R=x^2-x+1$, *and* $S=x^2+2x+1$, *find the following:*

28. $2K-(R+S)$. **29.** $3R-2(2K-S)$.

37. Placing terms within a parenthesis.

If $-(6ab+ac)$ is the same as $-6ab-ac$, then, when the process is reversed, $-6ab-ac=-(6ab+ac)$.

Similarly, $a+b+c$ is the same as $-(-a-b-c)$.

Rule. *If an expression is placed within a sign of aggregation preceded by the — sign, the sign of each term of the expression must be changed.*

Exercise 27

1. Insert the last three terms of the expression $a^3 - 3a^2b + 3ab^2 - b^3$ in a parenthesis preceded by the $-$ sign.

2. Insert all the expression $a^2 - 2ab + b^2$ in a parenthesis preceded by a $-$ sign.

3. Indicate the difference by means of a $-$ parenthesis when $a + b$ is subtracted from $c - d$. Ans. $c - d - (a + b)$.

4. Similarly, indicate the difference by means of a $-$ parenthesis when $a^3 - 3a^2b + 3ab^2 - b^3$ is taken from $a^3 - b^3$. Remove the parenthesis and combine.

Insert the last three terms of each of the following in a parenthesis preceded by a — sign:

5. $a + b + c + d$.

6. $x + y - z - 5$.

7. $m - n + 2rs - k$.

8. $x^2 - y^2 + 2yz - z^2$.

9. $a^2 - b^2 - 2bc - c^2$.

10. $4x^2 - 9y^2 + 6yz - z^2$.

11. $a^2 - 2ab + b^2 - m^2 + 2mn - n^2$.

12. $2t - 3r + 2s + 5z$.

Remove the following parentheses, collect like terms, and then rewrite the expressions with the last two terms of each in a parenthesis preceded by a — sign:

13. $a - (b - c) - d$.

14. $x^2 - (y^2 + 2yz + z^2)$.

15. $23 - (a - b - c)$.

16. $24 - 2(a - b) + 3(b - c)$.

17. $2a + 3(a - 2b) - 4(2 + 3c) - 5z$

18. $100 - 3(2n + 5) + 2(n - r) - 7(5 - 2r)$.

19. $8(a - 2b) + 3(4b + c) - (3c - d)$.

20. Remove signs of aggregation and collect:
$-(x-2y+3z)+(2x+2y-z)-(x+y+z)$.

21. Insert the last four terms of $a^4-4a^3b+6a^2b^2-4ab^3+b^4$ in a parenthesis preceded by a $-$ sign; the last two.

MULTIPLICATION AND DIVISION

38. Order of factors. Compare the products of $2 \cdot 3$ and $3 \cdot 2$. Compare the products of $2 \cdot 3 \cdot 4$, $2 \cdot 4 \cdot 3$, $3 \cdot 2 \cdot 4$, $3 \cdot 4 \cdot 2$, $4 \cdot 2 \cdot 3$, and $4 \cdot 3 \cdot 2$.

Take the product of 2, 3, 4, and 5 in as many ways as possible and notice that the results are all the same.

Evidently the order in which arithmetical factors are taken does not affect their product.

Give different sets of values to the literal numbers a and b and their product $a \cdot b$ will always be found equal to $b \cdot a$. In like manner $x \cdot y \cdot z = x \cdot z \cdot y = y \cdot x \cdot z = z \cdot x \cdot y$, etc.

Rule. *The factors of a product may be taken in any order without changing the result.*

39. The sign of the product. What is the sign of the product of $2 \cdot -3$? (Read this "2 times negative 3.") of $-2 \cdot 3$? of $-2 \cdot -3$? of $-2 \cdot -3 \cdot 4$? of $-2 \cdot -3 \cdot 4 \cdot 5$? of $-2 \cdot -3 \cdot -4$? of $-2 \cdot -3 \cdot -4 \cdot -5$? of $-2 \cdot -3 \cdot -4 \cdot -5 \cdot -6$?

We notice that if one or three factors of a group of factors are $-$, the sign of the product is $-$. If two or four factors are $-$, the sign of the product is $+$. Hence the following:

Rule. *The sign of the product is $+$ if an even number of its factors are $-$, and the sign of the product is $-$ if an odd number of its factors are $-$.*

40. Exponents in multiplication. Define "**exponent.**" How many times is a used as a factor in a^3? in a^5?

Note. Where no exponent is written the exponent is assumed to be 1, a^1 is the same as a.

The product of $a^3 \cdot a^2$ is the same as $(a \cdot a \cdot a) \cdot (a \cdot a)$, or a taken five times as a factor, written a^5.

Similarly, the product of $a^m \cdot a^n$ is $(a \cdot a \cdot a \cdots$ to m factors) times $(a \cdot a \cdot a \cdots$ to n factors), or $a \cdot a \cdot a \cdot a \cdots$ to $m + n$ factors, written a^{m+n}.

Typeform I. $a^m \cdot a^n \equiv a^{m+n}$.

The use of the most general exercise as a "**typeform,**" or **formula,** is a convenient method of stating a rule.

What is the product of $a^2 \cdot a^3 \cdot a^4$? of $a^2 \cdot a^3 \cdot a^4 \cdot a^5$?

The product of $a^2b^3 \cdot a^3b^4 \cdot a^3b^2$ is the same as $(a \cdot a \cdot b \cdot b \cdot b \cdot a \cdot a \cdot a \cdot b \cdot b \cdot b \cdot b \cdot a \cdot a \cdot a \cdot b \cdot b)$, in which a is taken $2 + 3 + 3$, or 8 times as a factor, and b, $3 + 4 + 2$, or 9 times as a factor. Therefore the required product is a^8b^9.

Rule. *The exponent of each literal number in a product of monomials is equal to the sum of all the exponents of that number in the factors.*

41. Exponents in division. Division is the inverse of multiplication. Since $a^2 \cdot a^5 \equiv a^7$, then $a^7 \div a^2 \equiv a^5$.

$a^7 \div a^2$ may also be written $\dfrac{aaaaaaa}{aa}$. If the common factors are divided out of numerator and denominator, $aaaaa$, or a^{7-2}, which is a^5, remains.

Typeform II. $a^m \div a^n \equiv a^{m-n}$.

Divide a^3b^2 by ab^2. Ans. a^2. Divide x^5y^6 by x^3y^4. Ans. x^2y^2.

Rule. *The exponent of each literal number in a quotient of monomials is equal to its exponent in the dividend minus its exponent in the divisor.*

Exercise 28

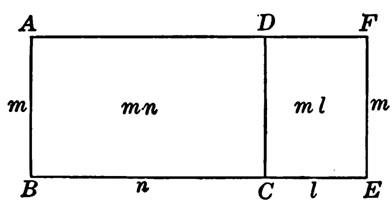
Find the following products or quotients:

1. $2a^2b^3c^4\cdot 3b^2c^2\cdot 4a^4c^3\cdot -ab^2c$.
2. $-3x^3y^3z^3\cdot 2x^2y^2z^2\cdot -4xyz$.
3. $2a^2b^2c^3d\cdot 3a^3b^3c\cdot -3ac^2d^2$.
4. $-2a^2bc\cdot -4ab^2d\cdot -3bc^3d$.
5. $-9m^2x\cdot 3mx^3\cdot -2mnx^4\cdot -3n^2x$.
6. $-2a^2\cdot -3b^2\cdot -4c^2\cdot -abc^2\cdot -ac$.
7. $3x\cdot 2x\cdot -x\cdot -5x\cdot -ax\cdot -bx$.
8. $2^2 3^2 5^2\cdot 2^3 3^3 5\cdot 2^2 5^5$. Ans. $2^7 3^5 5^8$.
9. $12a^3b^2c^2 \div -2a^2b^2c$.
10. $-8abc\cdot 2ac \div -4ab$.
11. $-4a^2\cdot -5a^3 \div -10a^4$.
12. $39a^2b^2c^3d^4 \div -3ab^2d$.
13. $30a^3b^3c^8 \div -6a^2b^3c$.
14. $2^2 2^3 2 \div 2^4$. Ans. 2^2.
15. $2^4 3^4 5^4 \div 3^3 5^3$.
16. $2^2\cdot -3^2\cdot -4^3 \div 2\cdot -3$.

42. Multiplication of a polynomial by a monomial. What is the product of $2(a-b)$? (See § 20.)

Similarly, the product of $m(n+l) = mn + ml$.

The accompanying figure will serve to illustrate the meaning of such a product.



If m is the number of units in the width of the rectangle $ABCD$ and n the number in its length, its area is mn square units. Similarly the area of $DCEF$ is ml square units and the sum of the two, or rectangle $ABEF$, has an area $mn + ml$ square units. Therefore the product of $m(n+l) = mn + ml$.

Make a rectangular figure which will illustrate that $a(b+c+d) = ab + ac + ad$.

What is the product of $2a(3b+2c)$? Ans. $6ab+4ac$.

The following is a convenient form for such a product: (Check with $a=2$, $b=3$, and $c=4$.)

$$\begin{array}{ll} 3b+2c & = 9+\ 8=17 \\ 2a & = \quad\quad\ \ 4 \\ \hline 6ab+4ac & =36+32=68. \end{array}$$

Try several other sets of numbers for checking this product. As the algebraic exercises increase in difficulty, a check of some kind will become a necessity. In general the student will be able to check his work by substituting in the given expressions a convenient set of values for the literal numbers, performing on the results the same operation as required in the exercise, and then substituting the same set of values in the answer of the exercise. If the results are the same from both operations, $(68=68)$, we say that the work "**checks.**"

Rule. *In multiplying a polynomial by a monomial, multiply each term of the polynomial by the monomial and write the successive products in a new polynomial, each with its proper sign.*

Exercise 29

Perform the following multiplications and check in each:

1. $\begin{array}{l} 2+a \\ 2 \\ \hline \end{array}$ **2.** $\begin{array}{l} a+b \\ 3 \\ \hline \end{array}$ **3.** $\begin{array}{l} a+c \\ a \\ \hline \end{array}$ **4.** $\begin{array}{l} b+c \\ 2b \\ \hline \end{array}$

5. $\begin{array}{l} a+b \\ ab \\ \hline \end{array}$ **6.** $\begin{array}{l} x+y \\ xy \\ \hline \end{array}$ **7.** $\begin{array}{l} 2a+3b \\ 5b \\ \hline \end{array}$

8. $\begin{array}{l} a+2bc \\ ac \\ \hline \end{array}$ **9.** $\begin{array}{l} 2bc+a \\ ab \\ \hline \end{array}$ **10.** $\begin{array}{l} ab+bc \\ 2b \\ \hline \end{array}$

11. $(ab+ac)\cdot ac$. **12.** $(ab+ac)\cdot abc$.

13. $(3ab - 4ac) \cdot 2bc.$

14. $(3ab - 4ac) \cdot 3abc.$

15. $(3xy - 5yz) \cdot 5xyz.$

16. $(x - y + z) \cdot xyz.$

17. $\begin{array}{l} 32x - 3y + 4z \\ -2x \\ \hline \end{array}$

18. $\begin{array}{l} 4ab - 3cd + 2ad \\ -3ax \\ \hline \end{array}$

19. $\begin{array}{l} 4x^2y + 5xy^2 \\ 2yz \\ \hline \end{array}$

20. $\begin{array}{l} 3a^3 + 2a^2 \\ 4a \\ \hline \end{array}$

21. $\begin{array}{l} 3a^3b - 3ab^3 \\ a^2b^2 \\ \hline \end{array}$

22. $\begin{array}{l} 2a^2b - 3a^3c \\ -2a^2bc^2 \\ \hline \end{array}$

23. $\begin{array}{l} xy - xz + yz \\ -xyz \\ \hline \end{array}$

24. $\begin{array}{l} 2x - 3y + 4z \\ -2x \\ \hline \end{array}$

25. $\begin{array}{l} 2mn - 3m - 2n \\ -3mn \\ \hline \end{array}$

26. $(2xy - 3xz - 4yz) \cdot 2xyz.$

27. $(a^2 - 2ab + b^2) \cdot ab.$

28. $(x^2 + 2x + 1) \cdot 2x.$

29. $(a^2 + b^2 + c) \cdot abc.$

30. $(x^3 - x^2 + x - 1) \cdot x.$

31. $(a^3 + b^2 + c^2) \cdot abc.$

32. $(x^2 - x + 1) \cdot (-x).$

33. $(x^2 - xy + y^2) \cdot (-xy).$

34. $(x^3 - x^2y + xy^2 - y^3) \cdot xy.$

35. $(x^3 - x^2y + xy^2 - y^2) \cdot (-xy).$

36. $(2a^2b - 3ab^2) \cdot (-ab).$

37. $(4ab - cd) \cdot (-abcd).$

38. $(9ab^2c - 6a^2b^2c) \cdot abc.$

39. $(xy - yz + zw)(-xyzw).$

40. $(2a^2 - 2b^2 - 2c^2)3a^2b^2c^2.$

41. $(2a^2y - 3b^2c - 4c^2)2abcy.$

42. $(a^3 - a^2b + ab^2 - b^3)(-a^2b^2).$

43. $(a^3b - a^2b^2 + ab^3 - b^4)(-ab^2).$

44. $\begin{array}{l} 2^2 - 3^2 \\ 2^23 \\ \hline \end{array}$

Ans. $2^43 - 2^23^3.$

45. $\begin{array}{l} 2^2 + 3^3 + 4^4 \\ 2^23^34^4 \\ \hline \end{array}$

46. $\begin{array}{l} 2^2 - 3^3 - 5^5 \\ 2^23^25^2 \\ \hline \end{array}$

43. Division of a polynomial by a monomial.

Division is the **inverse of multiplication,** therefore the **Rule.** *In dividing a polynomial by a monomial, divide each term of the dividend by the divisor and write the partial quotients, each with its own sign, in a new polynomial.*

The following is a convenient form for the work with its check, using $a=2$, $b=3$, $c=4$, and $d=1$.

$$\begin{array}{l} 2a)\underline{6ab + 4ac - 10ad} = 36 + 32 - 20 = 48. \quad 2a = 4 \\ \quad\;\; 3b + 2c - 5d = 9 + 8 - 5 = 12. \quad 48 \div 4 = 12. \end{array}$$

Exercise 30

Divide the following and check in each:

1. $(4ab - 6ac)$ by 2.
2. $(4ab - 6ac)$ by a.
3. $(4ab - 6ac)$ by $2a$.
4. $(8ab + 12ac)$ by $4a$.
5. $(2abc + 3acd)$ by ac.
6. $(4a^2 - 6a)$ by $2a$.
7. $(4a^2b - 6ab^2) \div (-2ab)$. Ans. $-2a + 3b$.
8. $\dfrac{6ab + 9ac}{3a}$.
9. $\dfrac{4a^3 - 6a^2}{2a}$.
10. $\dfrac{9a^2b - 12ab^2}{3ab}$.
11. $\dfrac{12ac - 15cd + 6acd}{3c}$.
12. $\dfrac{10bc - 20cd - 6c^2}{2c}$.
13. $\dfrac{9abc - 12acd + 15abcd}{3ac}$.
14. $\dfrac{2a^3b - 4a^2b^2 - 6ab^3}{2ab}$.
15. $(30xy - 35xz + 25x)$ by $5x$.
16. $(15a^4 - 25a^3 + 30a^2)$ by a^2.
17. $(4a^2bc + 10ab^2cd)$ by $2abc$.
18. $(15x^2yz - 18xy^2z)$ by $3xyz$.
19. $(4ab - 6ac)$ by $-a$.
20. $(4ab - 6ac)$ by $-2a$.
21. $(4x^2 - 8x^3)$ by $-2x^2$.

22. $(15m^3 - 20m^2)$ by $-5m^2$.

23. $(10a^2b^3c - 15ab^2c^2 + 25abc^3)$ by $5abc$.

24. $(24x^2y^3z^4 + 27x^3y^2z + 18xy^2z^3)$ by $-3xy^2z$.

25. $\dfrac{30a^2b^2x - 36ab^2x^2 + 48a^3x^2}{-6ax}$.

26. $\dfrac{x^5 - x^4 + x^3 + x^2}{-x^2}$.

27. $\dfrac{-16m^2n^3 - 24m^3n^2 - 8m^4n^3l^2}{-8m^2n^2}$.

28. $\dfrac{-4A^4 + 6A^3 - 8A^2}{2A^2}$.

29. $(30x^3y^5z^4 + 12x^5y^4z^3 - 18x^3y^3z^4)$ by $6x^3y^3z^3$.

30. $(-45a^3b^5 - 60a^4b^4 - 75a^5b^3)$ by $-15a^3b^2$.

31. $[8(a-b)^3 - 6(a-b)^2 - 4(a-b)]$ by $2(a-b)$.

32. $[-12x^2y(x-y)^2 + 9x(x-y)]$ by $-3x(x-y)$.

33. $[(a+b)^4 - 3(a+b)^3 + 3(a+b)^2 - (a+b)]$ by $(a+b)$.

44. Factoring. What are the factors of $3ab$? Ans. 3, a, and b. Of $12xyz$ when factored completely? Ans. 2, 2, 3, x, y, and z. Of $24a^2b^2c$? of $72k^3m^3n^2$? of $54R^3r^2$?

What are the factors of $ax + ay$?

From a study of the products obtained so far it is evident that the product of a binomial and a monomial is a new binomial with the monomial multiplier present in each term. In the expression $ax + ay$, inspection shows a as the monomial multiplier, and the factors of $ax + ay$ are a and $(x + y)$.

Similarly, the factors of $2ab + 4ay$ are $2a$ and $(b + 2y)$.

Ex. 1. Divide $14a^2x - 21ax^2$ by $7ax$.

Ex. 2. What are the factors of $14a^2x - 21ax^2$?

Exercise 31

Factor each of the following:

1. $ab + ac$.
2. $2ac - 2ad$.
3. $2xy - 4xz$.
4. $abc + 2abd$.
5. $5xz - 15xw$.
6. $a^2 + ab$. Ans. a and $(a + b)$.
7. $x^2y - xy^2$.
8. $3xyz + 6xyw$.
9. $15h^2 - 20hk$.
10. $5xy - 10x^2$.
11. $ab - ac + ad$.
12. $4ad - 6bd + 8d^2$.
13. $a^2bc + ab^2c$.
14. $x^2y - x^2y^2 + xy^2$.
15. $2x^2 + 4x^3 - 10x^4 - 6x^5$.
16. $8a^2 + 12ab^2 + 4ab^3$.
17. $5m^3 - 10m^2n + 5mn^2$.

45. Arrangement. A polynomial is said to be **arranged** with respect to any literal number when the exponents of that number either increase or decrease in successive terms. $x^3 + 2x^2 - x - 3$ is arranged in **descending** powers of x and $1 + a + a^2 + a^3$, in **ascending** powers of a.

Exercise 32

Arrange the following in descending powers:

1. $-x^3 + 1 + 3x + x^4 - 2x^2$.
2. $a^3 - 1 - a - a^2 + a^4$.
3. $3x^2 - x^3 + 2x - x^4 + 1 + x^5$.
4. $3n - 2n^2 + 5n^4 - 1 - 7n^3$.

Arrange the following in ascending powers of a:

5. $x^3 + a^3 + 3ax^2 + 3a^2x$.
6. $a^4 - a^3 + a^2 - a + 1$.
7. $a^5 - x^5 + ax^4 + a^3x^2 - a^2x^3 - a^4x$.
8. $a^4 + 6a^2b^2 + b^4 - 4ab^3 - 4a^3b$.

46. Multiplication of a polynomial by a polynomial.

A convenient arrangement of the work in finding the product of $x+2$ and $x+3$ and the checking is:

$$
\begin{array}{lll}
 & & \text{Check} \\
\text{Ex. 1.} & x+3 & =5+3=8 \\
 & \underline{x+2} & =\underline{5+2=7} \\
 & x^2+3x & 56 \\
 & \underline{2x+6} & \\
 & x^2+5x+6 & =25+25+6=56.
\end{array}
$$

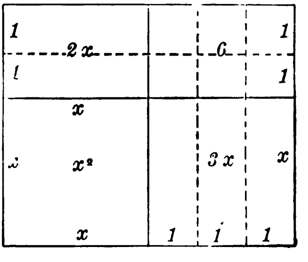
Since to obtain any product we must multiply all of the multiplicand by all of the multiplier, therefore to obtain the product of $x+3$ and $x+2$, we multiply $x+3$ by x, then by 2 and combine the results.

$$
\begin{array}{ll}
\text{Ex. 2.} & 30+2 \\
 & \underline{20+5} \\
 & 600+40 \\
 & \underline{150+10} \\
 & 600+190+10=800
\end{array}
$$

Notice the process for obtaining the product of 32 and 25, considering each as a binomial.

By a similar process find the product of 36 and 45.

Rule. *To multiply a polynomial by a polynomial, multiply each term of the multiplicand by each term of the multiplier and combine the results.*



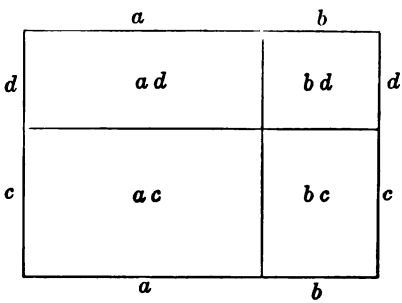
The accompanying figure may serve to explain how $(x+3)$ times $(x+2)$ can equal x^2+5x+6.

Make a similar figure for $(a+3)\ (a+4)$.

Exercise 33

Find the following products:

1. $(x + 4)(x + 5)$.
2. $(x + 5)(x + 6)$.
3. $(x - 2)(x - 3)$.
4. $(x - 4)(x - 5)$.
5. $(a - 7)(a - 3)$.
6. $(x - 4)(x + 5)$.
   Ans. $x^2 + x - 20$.
7. $(x - 30)(x + 20)$.
8. $(a + b)(a + b)$.
   Ans. $a^2 + 2ab + b^2$.
9. $(a - b)(a - b)$.
10. $(a - b)(a + b)$.
11. $(x + 10)(x - 12)$.
12. $(x + 12)(x - 10)$.
13. $(2x + 3y)(2x + 3y)$.
14. $(2x + 3y)(2x - 3y)$.
15. $(2x - 3y)(2x - 3y)$.
16. $(a + b)(a + c)$.
    Ans. $a^2 + ab + ac + bc$.
17. $(m + n)(m - k)$.



18. $(a + b)(c + d)$.
    Ans. $ac + bc + ad + bd$.
    (See figure.)
19. $(a + b + c)(a + b)$.
20. $(a + b + c)(a + b + c)$.
21. $(x^2 + 2x + 1)(x + 1)$.
22. $(x^2 + 2ax + a^2)(x + a)$.
23. $(a^2 - 2ab + b^2)(a - b)$.
24. $(x^3 + x^2 + 2x + 2)(x + 1)$.
25. $(x^3 + 3x^2 + 3x + 1)(x + 1)$.
26. $(c^3 - 3c^2 + 3c - 1)(c - 1)$.
27. $(2a^2x + 2ax^2)(a + x)$.
28. $(a^2b - ab^2)(a^2 - b^2)$.

29. $(-2 - x^2 + x + x^3)(2x - 1 + x^2)$.

Note: Arranging with reference to x gives:

$$
\begin{array}{lll}
x^3 - x^2 + x - 2 & & = 8 - 4 + 2 - 2 = 4 \\
\underline{x^2 + 2x - 1} & & = 4 + 4 - 1 \quad = \underline{7} \\
x^5 - x^4 + x^3 - 2x^2 & & 28 \text{ (product)} \\
\quad 2x^4 - 2x^3 + 2x^2 - 4x & & \\
\qquad \underline{- x^3 + x^2 - x + 2} & & \\
x^5 + x^4 - 2x^3 + x^2 - 5x + 2 = 32 + 16 - 16 + 4 - 10 + 2 = 28. & &
\end{array}
$$

Checking with $x = 2$.

Arrange terms before multiplying the following:

30. $(b^2 - b + b^3 - 2)(2 + b^2 - b)$.

31. $(3ab^2 - 3a^2b + a^3 - b^3)(a - b)$.

32. $(1 + 2m^2 - 2m - m^3)(2m + m^3 + 2m^2 + 1)$.

33. $(a^2 - b^2 - 3ab)(a^2 + b^2 + 3ab)$.

34. $(d^3 + 3d - 4d^2 + 1)(5 + d^2 - 2d)$.

47. Division of a polynomial by a polynomial.

Since division is the inverse of multiplication, compare the following process with that of § 46:

Divide $x^2 + 5x + 6$ by $x + 3$.

$$
\begin{array}{l}
x^2 + 5x + 6 \,)\, \underline{x + 3} \\
\underline{x^2 + 3x} \qquad)\, \underline{x + 2} \text{ Ans.} \\
\qquad 2x + 6 \\
\qquad \underline{2x + 6}
\end{array}
$$

Check with $x = 5$

dividend $= 56$
divisor $= 8$
quotient $= 7$.
$56 \div 8 = 7$.

Since x^2, the first term of the dividend, is the product of the first term of the divisor and the first term of the quotient, therefore the first term of the quotient is x. Multiplying $x + 3$ by x we obtain $x^2 + 3x$, which we recognize as one of the partial products of § 46. Subtracting this product and repeating the process, we obtain the next term of the quotient.

Rule. I. *Arrange the dividend and divisor in the same order of powers of some common literal number.*

II. *Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.*

III. *Multiply the whole divisor by the first term of the quotient, and subtract the product from the dividend.*

IV. *Arrange the remainder in the same order of powers as the dividend, consider it as a new dividend, and proceed as before.*

V. *Repeat the process until there is no remainder, or until no term of the remainder will contain the first term of the divisor.*

Exercise 34

Divide in each of the following and check the results numerically:

1. $a^2 + 6a + 8$ by $a + 2$.

2. $x^2 + 5x + 6$ by $x + 3$.

3. $m^2 + 7m + 12$ by $m + 4$.

4. $n^2 + 7n + 10$ by $n + 2$.

5. $a^2 - 5a + 6$ by $a - 3$.

6. $x^2 - 9x + 14$ by $x - 7$.

7. $r^2 - 8r + 15$ by $r - 3$.

8. $a^2 - 10a + 21$ by $a - 3$.

9. $x^2 + x - 12$ by $x + 4$.

10. $x^2 - x - 12$ by $x - 4$.

11. $m^2 - 2m - 15$ by $m + 3$.

12. $a^2 + 3a - 28$ by $a - 4$.

13. $a^2 + 2ab + b^2$ by $a + b$.

14. $a^2 - 2ab + b^2$ by $a - b$.

15. $m^2 - 6mn + 9n^2$ by $m - 3n$.

16. $r^2 + 11rs + 30s^2$ by $r + 5s$.

17. $x^2 - 9$ by $x - 3$.

18. $a^2 - b^2$ by $a - b$.

19. $n^2 - 25$ by $n + 5$.

20. $x^2 - y^2$ by $x + y$.

Inspect the following and write the quotients without placing the work on paper:

(Notice that in No. 1, $8 = 2 \cdot 4$ and $6 = 2 + 4$.)

21. $n^2 + 3n + 2$ by $n + 2$.

22. $a^2 + 5a + 6$ by $a + 2$.

23. $a^2 + 10a + 24$ by $a + 4$.

24. $r^2 + 12r + 36$ by $r + 6$.

25. $x^2 - 2x - 8$ by $x + 2$.

26. $x^2 + 5x - 24$ by $x - 3$.

27. $x^2 - x - 20$ by $x - 5$.

28. $n^2 - 16$ by $n + 4$.

29. $m^2 - 3m - 40$ by $m - 8$.

30. $4 - b^2$ by $2 + b$.

Arrange when necessary in descending powers of the same literal number and divide:

(The results should be proved by checking or by multiplying the divisor by the quotient.)

31. $6a^2 - 4a^3 - 4a + a^4 + 1$ by $a^2 + 1 - 2a$.

Solution. Arranging in descending powers of a:

$$
\begin{array}{r|l}
a^4 - 4a^3 + 6a^2 - 4a + 1 & a^2 - 2a + 1 \\ \cline{2-2}
\underline{a^4 - 2a^3 + a^2 \qquad\qquad} & a^2 - 2a + 1 \\
-2a^3 + 5a^2 - 4a \qquad & \\
\underline{-2a^3 + 4a^2 - 2a \qquad} & \\
a^2 - 2a + 1 & \\
\underline{a^2 - 2a + 1} &
\end{array}
$$

Checking with $a = 3$,

dividend $= 81 - 108 + 54 - 12 + 1 = 16$

divisor $= 9 - 6 + 1 = 4$

quotient $= 9 - 6 + 1 = 4$.

$16 \div 4 = 4$.

32. $a^3 - b^3$ by $a - b$.

Solution.

$$\begin{array}{l|l}
a^3 - b^3 & a - b \\ \cline{2-2}
a^3 - a^2b & a^2 + ab + b^2 \\ \cline{1-1}
\quad a^2b - b^3 & \\
\quad a^2b - ab^2 & \\ \cline{1-1}
\qquad ab^2 - b^3 & \\
\qquad ab^2 - b^3 & \\ \cline{1-1}
\end{array}$$

Checking with $a = 5$, $b = 2$,
dividend $= 125 - 8 = 117$
divisor $= 5 - 2 = 3$
quotient $= 25 + 10 + 4 = 39$.
$117 \div 3 = 39$.

33. $3a^2 + 3a + a^3 + 1$ by $a + 1$.

34. $3x - 3x^2 + x^3 - 1$ by $x - 1$.

35. $3a^2b + 3ab^2 + a^3 + b^3$ by $a^2 + b^2 + 2ab$.

36. $8x^2 - 22xy + 15y^2$ by $2x - 3y$.

37. $6x^2 - 26x + 28$ by $3x - 7$.

38. $6a^2 + 6b^2 + 13ab$ by $3b + 2a$.

39. $1 + c^3 + 2c + 2c^2$ by $c + 1$.

40. $8 + c^3 + 12c + 6c^2$ by $c + 2$.

41. $a^3 + b^3$ by $a + b$. **42.** $x^3 - 8$ by $x - 2$.

43. $27a^3 - 8b^3$ by $3a - 2b$.

44. $a^4 - b^4$ by $a - b$. **45.** $a^4 - b^4$ by $a + b$.

46. $12x^2 - 25xy + 12y^2$ by $4x - 3y$.

47. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ by $x^2 + 2xy + y^2$.

48. $n^4 + 2n^3 + 2n^2 + n - 2$ by $n^2 + n - 1$.

49. $x^4 + 2x^3y - 6x^2y^2 + 26xy^3 - 15y^4$ by $x^2 + 4xy - 3y^2$.

50. $a^4 + a^2 + 1$ by $a^2 + a + 1$.

51. $4a^3 - a^2 + 4a^4 - 2 + 7a$ by $2 + 2a^2 - a$.

52. $x^2 - 2xy + y^2 - z^2$ by $x - y - z$.

53. $a^2 - b^2 - 2bc - c^2$ by $a + b + c$.

54. $4x^2 - 9y^2 + 12yz - 4z^2$ by $2x + 3y - 2z$.

48. Factoring trinomials. The two binomial factors of certain trinomials may be obtained by inspection.

In x^2+5x+6, $6=2\cdot 3$ and $5=2+3$, therefore its factors are $x+2$ and $x+3$. The factors of x^2-5x+6 are $x-3$ and $x-2$. Test by taking their product.

Similarly in factoring x^2+x-6, notice that -6 equals either $-3\cdot 2$ or $3\cdot -2$, but $+x$ requires that the $+$ factor of 6 be 1 larger than the $-$ factor, therefore $x^2+x-6=(x+3)(x-2)$. Check by multiplication.

Exercise 35

Factor the following by inspection and check by multiplication:

1. x^2+3x+2.
2. x^2-3x+2.
3. x^2+x-2.
4. x^2-x-2.
5. a^2+7a+6.
6. a^2-5a+6.
7. a^2+5a-6.
8. a^2+a-6.
9. b^2+9b+8.
10. b^2-7b-8.
11. b^2-2b-8.
12. b^2-6b+8.
13. $y^2+8y+15$.
14. $y^2+2y-15$.
15. $y^2-14y-15$.
16. $y^2-2y-15$.
17. y^2+y-20.
18. $y^2-8y-20$.

Exercise 36. Review

Simplify the following:

1. $13-3(5-3)+2(3+4)-3(-3-4)$.
2. $2(8-5)-3(-3+6)+2(2-2-7)-18$.
3. $5a-3(2a+4b)+2(3a-2b)-27$.
4. $(2x+3)(3x-2)-2(2x-1)(x+4)$.

Note. First obtain the products before removing the parentheses entirely. The second step of the work should be:
$(6x^2+5x-6)-2(2x^2+7x-4)$. Ans. $2x^2-9x+2$.

5. $3(x-1)(2x+3)+5(3x-2)$.

6. $2(2a-3)(3a+1)+2(a+2)(4a-3)$.

7. $(2n+7)(n-2)-(n+5)(n-4)$.

8. $2(x+1)(x+2)+3(x-5)(x+3)-4(x+2)$.

9. $3(a-2)(a+2)-3(a-1)(a-1)+2(a+1)$.

10. $(m^2-2mn+n^2)-3(m-n)(m+n)$.

11. $2a(a+b)-3b(a-b)-(a-b)(a+b)$.

Solve the following equations and check:

12. $13-3(x+2)-2(2x+5)=2x+15$.

13. $3a+2(6-a)-3(2a+3)=18$.

14. $2(2n+5)-4(n+2)=8-3(n+4)$.

15. $(x-2)(x-9)-7=(x-2)(x+5)-(x+5)$.

Note. Follow the plan of No. 4 for obtaining the products and removing the parentheses.

16. $(x+5)(x-5)=(x-4)(x+3)$.

17. $(n+7)(n-3)-n^2=3n-16$.

18. $(a+5)(a-2)+(a-3)(a+4)=2a(a+3)-2$.

19. $(n+4)(n-2)+n(n-3)=2(n+5)(n-8)+7$.

20. $(x-8)(x+3)-(x+4)(x-2)=4-9x$.

21. $(m+6)(m-5)-(m-3)(m-4)=2-3m$.

22. $(2n+1)(n-5)-(n+1)(n-7)=n(n-4)$.

23. $(2k+3)(2k-3)-(4k+1)(k-5)=3(k-4)$.

24. $(2a+7)(3a-11)-(6a-9)(a-3)=0$.

25. $(2n+1)(2n+1)-4(n-3)(n+7)=15n+4$.

26. $3(a+1)(4a-2)-2(3a-1)(2a+1)=0$.

27. $2(x+1)(x-2)-x^2=x^2-2(3x+8)$.

28. $5n^2+3(2-n)(3+n)-2(n+4)(n-3)=2$.

29. $(r+5)(r-7)-3(3-r)=(r-4)(r-1)$.

30. $25-3(x+2)-(5-x)(5+x)=x^2+9$.

If $a=2$, $b=3$, and $c=-4$, evaluate the following:

31. $a^3+3a^2b+3ab^2+b^3$.

32. $a^3-3a^2b+3ab^2-b^3$.

33. $a^3+b^3+c^3-3abc-ab-ac-bc$.

34. $a^2-b^2-c^2+2b^2c-3bc^2$.

35. Factor completely: $16a^3b^2$; $48a^2b^2c$; $3x+3y$; and $6x^2-9xy$.

36. From the sum of x^2-2x+3 and $2x^2+4x-7$ take x^2-4x-2.

37. Subtract $7ab+3ac-2bc-5$ from $3ab-ac-bc$.

38. From $3a^2-5ab-b^2$ take $4a^2+ab-b^2-8$.

Find the remainder in each of the following by direct subtraction, and also by removing the parentheses and combining like terms:

39. $(2x-3y+5z)-(4x+7y-3z)$.

40. $(2xy-3xz+4yz-3)-(6xy+2xz-7yz-11)$.

41. $(13+2mn-3mr+6nr)-(4mn-11+2nr+6mr)$.

42. $(x^4-3x^2+5x^3-x-8)-(4+x^2-2x^4+3x^3+11x)$.

Find the following products:

43. $(2n^2-3n+2)(3n^2+n-1)$.

44. $(a^2-5ab-b^2)(a^2+5ab+b^2)$.

45. $(x+2y-3z)(x-2y+3z)$.

46. $(x^2-xy+x+y^2+y+1)(x+y-1)$.

47. $(a^2+b^2+c^2-ab-ac-bc)(a+b+c)$.

Divide the following:

48. $4x^3-4x^2-7x+6$ by $2x-3$.

49. $n^4-n^3-3n^2+8n-6$ by n^2+n-3.

50. $8a^4 - 22a^3 + 43a^2 - 38a + 24$ by $4a^2 - 5a + 6$.

51. $2x^4 + 2y^4 + 4xy^3 + x^3y - 11x^2y^2$ by $x^2 - 2y^2 + 2xy$.

Suggestion. Arrange dividend and divisor in the order of the descending powers of x.

52. $15a^4 - a + 8a^2 - 1 - 19a^3$ by $5a^2 - 1 - 3a$.

53. $c^4 - c^3 - 10 + 16c - 3c^2$ by $c^2 - 2 + 2c$.

54. $m^3 + 3mn + n^3 - 1$ by $m + n - 1$.

Solve the following problems:

55. Find three consecutive numbers whose sum is 45.

56. Find three consecutive even numbers such that the first plus 3 times the second minus twice the third equals 26.

Suggestion. If n, $n + 2$, and $n + 4$ represent the numbers, then $n + 3(n + 2) - 2(n + 4) = 26$. Study carefully the use of the parentheses.

57. The sum of two numbers is 15, and 3 times the first minus 2 times the second equals 20. Find the numbers.

Suggestion. Let n and $15 - n$ represent the numbers. How do you write 2 times $15 - n$?

58. John and James together have 38 cents. If twice the amount that John has be added to 3 times the amount that James has, the result is 99 cents. How much has each?

59. A sum of money, amounting to $1.60, consists of nickels and quarters. If there are 12 coins altogether, how many are there of each kind?

60. A is twice as old as B, and 15 years ago he was 3 times as old. Find their present ages.

Suggestion. Get numerical expressions for their ages now and also for their ages 15 years ago.

61. A is 4 times as old as B, and in 6 years he will be only 3 times as old. What are their ages?

62. The second of two numbers is 2 more than 3 times the first. If the second be subtracted from 5 times the first, the remainder is 10. Find the numbers.

63. The second of three numbers is 1 more than twice the first, and the third is 1 less than 3 times the first. Find the numbers if their sum is 42.

64. The total number of pupils in two algebra classes is 43. If twice the number in the second class be subtracted from 3 times the number in the first class, the remainder is 34. Find the number in each class.

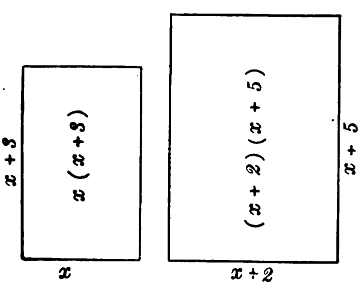
65. Find three consecutive numbers such that 3 times the first plus the second minus twice the third equals 11.

66. Find four consecutive numbers such that the product of the first and second is 26 less than the product of the third and fourth.

Suggestion. Using n, $n+1$, $n+2$, and $n+3$ to represent the numbers, the equation is $n(n+1)+26=(n+2)(n+3)$.

67. Find three consecutive numbers such that the square of the smaller is 32 less than the product of the other two.

68. A rectangle is 3 inches longer than it is wide. Another rectangle, whose dimensions are each 2 inches more than those of the first rectangle, has 30 square inches more in its area. Find the dimensions of each rectangle.



Suggestion. Get numerical expressions to represent the width, length, and area of each rectangle. One way of writing the equation is, $(x+2)(x+5)=x(x+3)+30$.

69. A rectangle is 5 inches longer than it is wide. If the width is increased 2 inches and the length is increased 1 inch, the area will be increased 30 square inches. Find the dimensions.

70. Charles has a garden that is 5 feet longer than it is wide. He finds that if he increases the width and length each 10 feet, there will be 650 square feet more in the area. Find the dimensions of the garden.

71. A rectangle is 7 inches longer than wide. If the length is decreased 4 inches and the width is increased 3 inches, the resulting rectangle will have the same area as the old. Find the dimensions of each rectangle.

72. A rectangular garden that is 20 feet longer than wide has a border 5 feet wide entirely around it. If the area of the border alone is 1,100 square feet, find the dimensions of the garden.

73. A group of n boys paid $n + 5$ cents each for the use of a boat and a group of $n - 2$ boys paid $n + 15$ for the use of a second boat that cost 50 cents more than the first. Find the number of boys in each group, what each paid, and the cost of each boat.

74. A number of men agreed to share equally in hiring a private railway car for an excursion. They discovered that the number of dollars each one owed for the car was 6 more than the number of men in the party. On another occasion, when the number of men was 4 less than before and the entire expense the same, each man paid $8 more than he did the first time. Find the number of men in each party, what each paid, and the total expense.

75. The second of two numbers is 5 less than three times the first. If the first is subtracted from 40 and the second from 49, the remainders are equal. Find the numbers.

76. The sum of two numbers is 16. If 5 times the second is subtracted from 50, the remainder is 2 times the first. What are the numbers?

77. Separate the number 40 into two parts such that the sum of 3 times one part and 5 times the other part may be 150.

78. The sum of the ages of A and B is 82 years. Five years ago A was 3 times as old as B. Find their present ages.

79. The sum of two numbers is 13, and one of them with 30 added to it is equal to 3 times the sum of the other number and 5. Find the two numbers.

80. There are three numbers. The sum of the first two is 12 and the third exceeds the first by 3. If the product of the first and second is added to the product of the first and third, the sum is 75. Find the three numbers.

81. Write an algebraic expression for the number of square feet in the walls and ceiling of a room that is m feet long, n feet wide, and k feet high.

82. A room is l feet long, w feet wide, and h feet high. It has two doors, b feet wide and a feet high, and 3 windows, b feet wide and c feet high. Find the number of square yards of plastering in the walls and ceiling of the room.

83. A man has m quarters, n dimes, and c cents. He buys b books that cost k cents each, and p pencils that cost 3 cents each. Write an algebraic expression for the number of cents he has left.

84. What is the area of a circle whose radius is 11 inches?

Hint. $A = \pi r^2$.

CHAPTER IV

SPECIAL PRODUCTS AND FACTORING

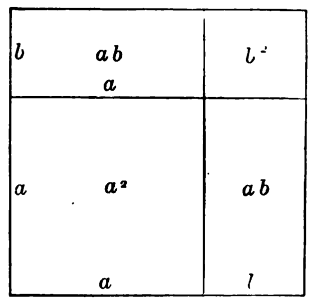
49. The square of the sum of two numbers.

What is the product of $(a + b)\ (a + b)$?

Expand the following by actual multiplication:

$(x + y)^2$; $(m + n)^2$; $(a + 2b)^2$; $(2a + 3b)^2$; $(5x + 7m)^2$.

Typeform $(a + b)^2 \equiv a^2 + 2ab + b^2$. The use of this typeform enables the student to write at sight the square of any binomial that is the sum of two terms.



This figure will serve to show the meaning of the typeform $(a + b)^2 \equiv a^2 + 2ab + b^2$. The large square consists of two small squares whose area are a^2 and b^2 and two rectangles whose areas are each ab.

Translated into words this typeform gives the following:

Rule. *The square of the sum of two numbers equals the square of the first plus twice the product of the first and second plus the square of the second.*

Exercise 37

Write at sight the following squares:

1. $(m + x)^2$.	**4.** $(2n + y)^2$.	**7.** $(5R + 7r)^2$.
2. $(m + 2)^2$.	**5.** $(3x + 2y)^2$.	**8.** $(3x + 10y)^2$.
3. $(2n + 1)^2$.	**6.** $(5A + 3B)^2$.	**9.** $(12C + 5D)^2$.

The following trinomials are squares of what binomials?

10. $x^2 + 6x + 9$.
11. $a^2 + 4a + 4$.
12. $x^2 + 8x + 16$.
13. $n^2 + 6n + 9$.
14. $9n^2 + 6n + 1$.
15. $a^2 + 12a + 36$.
16. $25x^2 + 10xy + y^2$.
17. $n^2 + 14ny + 49y^2$.
18. $9a^2 + 30ab + 25b^2$.
19. $81A^2 + 36AB + 4B^2$.

Complete the following trinomial squares by supplying the missing terms:

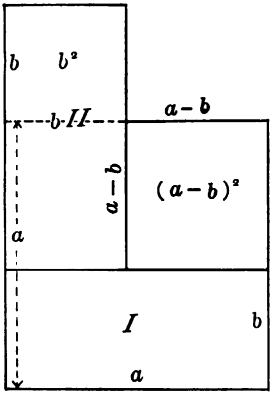
20. $x^2 + 6x + ?$
21. $x^2 + 8x + ?$
22. $m^2 + 2mn + ?$
23. $A^2 + ? + 16$.
24. $4x^2 + ? + 9$
25. $R^2 + ? + 25r^2$.

50. The square of the difference of two numbers.

What is the product of $(a - b)\ (a - b)$?

Expand the following by actual multiplication:
$(x - y)^2$; $(m - n)^2$; $(2a - b)^2$; $(2a - 3b)^2$; $(5a - 3)^2$.

Typeform $(a - b)^2 \equiv a^2 - 2ab + b^2$ enables the student to write at sight the square of any binomial that is the difference of two terms.



The accompanying figure will help to explain this second typeform, $(a - b)^2 \equiv a^2 - 2ab + b^2$. The square whose side is $a - b$ is formed from the sum of the two squares whose sides are a and b by taking away the two rectangles *I* and *II*, each of whose areas is ab.

Translated into words this typeform gives the following:

Rule. *The square of the difference of two numbers equals the square of the first minus twice the product of the first and the second plus the square of the second.*

Exercise 38

Write at sight the following squares:

1. $(m-5)^2$. 2. $(2x-1)^2$. 3. $(3x-2)^2$.
4. $(3a-2b)^2$. 5. $(5x-7y)^2$. 6. $(4m-5n)^2$.
7. $(5R-7r)^2$. 8. $(10A-3B)^2$.
9. $(12M-5N)^2$. 10. $(2ab-3c)^2$.

The following trinomials are squares of what binomials?

11. $m^2-2mn+n^2$.
12. $9-6x+x^2$.
13. a^2-6a+9.
14. $100x^2-20x+1$.
15. $16C^2-24CD+9D^2$.
16. $4N^2-12N+9$.
17. $36R^2-48Rr+16r^2$.
18. $64K^2-48Kh+9h^2$.

Complete the following trinomial squares by supplying the missing terms:

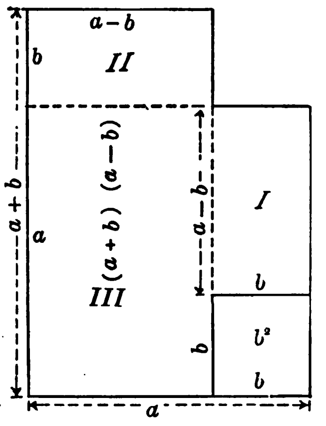
19. $m^2-10mn+$?
20. ? $-28mn+4m^2$.
21. b^2- ? $+36a^2$.
22. $4n^2-$? $+36x^2$.
23. $4x^2-4x+$?
24. $4n^2-12n+$?
25. ? $-8n+16$.
26. ? $+10n+n^2$.
27. $9a^2+$? $+36b^2$.
28. $4n^2+20an+$?
29. $9a^2b^2-$? $+1$.
30. $x^2y^2-4abxy+$?
31. $4y^4-$? $+x^2$.
32. $16a^4+32a^2+$?
33. $9x^2a^4+$? $+4b^2$.
34. ? $+12ab+b^2$.

51. The product of the sum and the difference of two numbers.

What is the product of $(a+b)(a-b)$?

Expand the following by actual multiplication:
$(x+y)(x-y)$; $(M+N)(M-N)$; $(2a+b)(2a-b)$; $(n+4)(n-4)$; $(3x+2y)(3x-2y)$; $(5r+7)(5r-7)$.

Typeform $(a + b)(a - b) \equiv a^2 - b^2$ enables the student to write at sight the product of the sum and the difference of any two numbers.



The accompanying figure will help to explain the typeform $(a + b)(a - b) = a^2 - b^2$. If we cut away from the large square whose side is a, the small square whose side is b, we will have left the two rectangles I and III. Now if I is cut away and placed in the position marked as II, then the long rectangle formed by II and III placed end to end has a width $a - b$ and a length $a + b$. Its area is $a^2 - b^2$.

Translated into words this typeform gives the following:

Rule. *The product of the sum and the difference of two numbers equals the difference of their squares.*

Exercise 39

Write at sight the following products:

1. $(n + 5)(n - 5)$.	**6.** $(6m + 7n)(6m - 7n)$.
2. $(2n + 1)(2n - 1)$.	**7.** $(5ab - 1)(5ab + 1)$.
3. $(2n + 5)(2n - 5)$.	**8.** $(b - 4R)(b + 4R)$.
4. $(3a + 7b)(3a - 7b)$.	**9.** $(10m - 4n)(10m + 4n)$.
5. $(3 + 5s)(3 - 5s)$.	**10.** $(12m - 7n)(12m + 7n)$.

Can you tell by inspection what binomials were taken to form the following products?

11. $a^2 - k^2$.	**15.** $36R^2 - 4r^2$.
12. $4 - 9m^2$.	**16.** $16A^2 - 36B^2$.
13. $9a^2 - 4b^2$.	**17.** $100C^2 - 81D^2$.
14. $25 - 16n^2$.	**18.** $49x^2 - 25y^2$.

52. The product of two binomials having a common term.

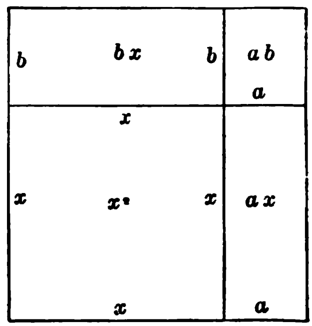
Find by direct multiplication the following products: $(x+3)(x+4)$; $(x-3)(x-4)$; $(x+3)(x-4)$; $(x-3)(x+4)$.

What is the product of $(x+a)(x+b)$?

Typeform $(x+a)(x+b) \equiv x^2+ax+bx+ab$

$$\equiv x^2+(a+b)x+ab.$$

The use of this typeform enables the student to write at sight the product of two binomials having a common term.



Study the figure.

Translated into words this typeform gives the following:

Rule. *The product of two binomials having a common term may be taken as a trinomial whose terms are,—*

(1) *The square of the common term,*

(2) *The product of the sum of the unlike terms and the common term, and*

(3) *The product of the unlike terms.*

Exercise 40

Write at sight the following products:

1. $(x+4)(x+2)$.
2. $(a+7)(a+2)$.
3. $(b-3)(b-4)$.
4. $(c-5)(c-2)$.
5. $(m+5)(m-4)$.
6. $(d-5)(d+6)$.
7. $(K-3)(K+9)$.
8. $(R-8)(R+3)$.
9. $(a-2)(a+7)$.
10. $(c-7)(c+4)$.
11. $(c-4)(c+7)$.
12. $(t-10)(t+8)$.

Write from inspection the binomials that were taken to form the following products:

13. $a^2 + 6a + 8$.	**21.** $c^2 - 5c - 14$.
14. $b^2 + 5b + 6$.	**22.** $x^2 - x - 12$.
15. $n^2 - 7n + 12$.	**23.** $x^2 + x - 12$.
16. $a^2 + 2a - 8$.	**24.** $x^2 - 4x - 12$.
17. $d^2 - d - 6$.	**25.** $a^2 - 8a + 15$.
18. $c^2 - 3c - 10$.	**26.** $a^2 - 2a - 15$.
19. $c^2 + 5c - 14$.	**27.** $a^2 - 16a + 15$.
20. $c^2 + 3c - 10$.	**28.** $a^2 - 14a - 15$.

53. The product of two binomials with similar terms.

Find the product of $(2x + 3y)(4x + 5y)$.

$$\begin{array}{l}
\quad 2x + 3y \\
\quad \underline{4x + 5y} \\
2\cdot 4x^2 + 4\cdot 3xy \\
\qquad\quad \underline{2\cdot 5xy + 3\cdot 5y^2} \\
2\cdot 4x^2 + (4\cdot 3 + 2\cdot 5)\,xy + 3\cdot 5y^2 = 8x^2 + 22xy + 15y^2.
\end{array}$$

Obtain by direct multiplication the following products:

$(2x + 3y)(3x + 2y)$; $(2x - 3y)(3x - 2y)$;
$(2x + 3y)(3x - 2y)$; $(2x - 3y)(3x + 2y)$;
$(2x - 3y)(2x + 3y)$; $(2x - 3y)(2x - 3y)$.

Note that each product is found according to the following:

Rule. *The product of two binomials with similar terms is a trinomial whose terms are,—*

(1) *The product of the first terms,*

(2) *The sum of the cross products (first term of first times second term of second plus second term of first times first term of second), and*

(3) *The product of the second terms.*

Exercise 41

Write by inspection the following products:

1. $(3x + 2y)(2x + y)$.
2. $(3x + 2y)(2x + 5y)$.
3. $(3x - 2y)(2x + 5y)$.
4. $(3x + 2y)(2x - 5y)$.
5. $(3x + 2y)(4x + 3y)$.
6. $(2m + 3n)(3m - 5n)$.
7. $(4a + 3b)(2a + 5b)$.
8. $(4a - 3b)(5a + 2b)$.
9. $(3m + 4n)(2m - 7n)$.
10. $(5R - 2s)(3R + 4s)$.
11. $(3h + 2r)(5h - 6r)$.
12. $(2a - b)(a - b)$.
13. $(4c + 1)(2c + 3)$.
14. $(5r + 3s)(4r + 3s)$.

15. Can you tell what binomials were taken to form the product $2x^2 + 3x + 1$?

A study of the rule clearly indicates that we must find two binomials such that the product of their first terms shall be $2x^2$, and of their second terms, $+ 1$, and the sum of the cross products, $+ 3x$. $2x + 1$ and $x + 1$ seem to fill these requirements and their product gives $2x^2 + 3x + 1$.

It may be necessary to try out several sets of binomials before the right one appears. In factoring $2x^2 + x - 1$, it is difficult to tell by inspection whether the factors are $2x + 1$ and $x - 1$ or $2x - 1$ and $x + 1$. Only multiplication will serve to show which are the factors.

Factor the following:

16. $2x^2 - 3x + 1$.
17. $2x^2 - x - 1$.
18. $3x^2 + 4x + 1$.
19. $3x^2 - 4x + 1$.
20. $3x^2 + 2x - 1$.
21. $3x^2 - 2x - 1$.
22. $2x^2 + 5x + 3$.
23. $3x^2 + 5x + 2$.
24. $3x^2 + x - 2$.
25. $3x^2 - x - 2$.
26. $3x^2 - 5x + 2$.
27. $3x^2 + 8x + 4$.

54. Arithmetical products obtained at sight by the rules for special products.

Exercise 42

I. $45^2 = (40 + 5)(40 + 5) = 1600 + 400 + 25 = 2025.$

Quote the typeform used and apply it to the following, stating the product on inspection:

1. 21^2.	**6.** 41^2.	**11.** 44^2.	**16.** 71^2.
2. 23^2.	**7.** 42^2.	**12.** 65^2.	**17.** 72^2.
3. 31^2.	**8.** 43^2.	**13.** 53^2.	**18.** 81^2.
4. 32^2.	**9.** 33^2.	**14.** 52^2.	**19.** 92^2.
5. 35^2.	**10.** 51^2.	**15.** 61^2.	**20.** 102^2.

II. $27^2 = (30 - 3)(30 - 3) = 900 - 180 + 9 = 729.$

Quote the typeform and apply it to the following:

21. 28^2.	**25.** 59^2.	**29.** 78^2.	**33.** 57^2.
22. 29^2.	**26.** 58^2.	**30.** 68^2.	**34.** 67^2.
23. 38^2.	**27.** 69^2.	**31.** 89^2.	**35.** 77^2.
24. 49^2.	**28.** 79^2.	**32.** 99^2.	**36.** 88^2.

III. $32 \cdot 35 = (30 + 2)(30 + 5) = 900 + 210 + 10 = 1120.$
$23 \cdot 33 = (20 + 3)(30 + 3) = 600 + 150 + 9 = 759.$

Quote the typeform and apply it to the following:

37. $22 \cdot 32$.	**41.** $52 \cdot 62$.	**45.** $35 \cdot 45$.
38. $21 \cdot 24$.	**42.** $32 \cdot 36$.	**46.** $42 \cdot 46$.
39. $24 \cdot 34$.	**43.** $31 \cdot 61$.	**47.** $72 \cdot 52$.
40. $42 \cdot 45$.	**44.** $52 \cdot 53$.	**48.** $92 \cdot 102$.

IV. $27 \cdot 33 = (30 - 3)(30 + 3) = 900 - 9 = 891.$

Quote the typeform and apply it to the following:

49. $48 \cdot 52$.	**52.** $89 \cdot 91$.	**55.** $55 \cdot 65$.
50. $37 \cdot 43$.	**53.** $87 \cdot 93$.	**56.** $54 \cdot 66$.
51. $39 \cdot 41$.	**54.** $76 \cdot 84$.	**57.** $75 \cdot 85$.

FACTORING

55. Factoring monomials. Factoring in arithmetic consisted entirely in finding the factors of monomials. While there are many types for factoring in algebra, and the student must cultivate the ability to recognize quickly the various types, yet the simple monomial expression must not be overlooked. As we proceed with the more difficult polynomials, we must keep in mind that the factors of such numbers as $3a^2b$ may rank in importance with the factors of more difficult expressions.

56. The first five types for factoring polynomials have been introduced.

Type I. A polynomial with a monomial factor. (See § 44.)

Exercise 43

Write at sight all the factors of each of the following expressions. Check by multiplying the factors together, if the accuracy of the work is in doubt.

1. $3a + 3b$.

2. $ax + bx$.

3. $2a + 4b$.

4. $xy - 2x$.

5. $7xy - 14xz$.

6. $a^2 - ab$. Ans. a and $(a - b)$.

7. $2a^2 - 2ax$.

8. $a^3 - 2a^2$.

9. $5x^2 - 10xy$.

10. $3ax - 6axy$. Ans. 3, a, x, and $(1 - 2y)$.

11. $3A^2B + 9AB^2$.

12. $c^3y + 3c^2y$.

13. $12a^2bc + 18ab^2c$. Ans. 2, 3, a, b, c, and $(2a + 3b)$.

14. $2nR^2 + 2nR \cdot H$.

15. $3^2t^3 + 3st^2$.

16. $3x - 3y + 3z$.

17. $ax - ay + az$.

18. $a^3 - 2a^2x + ay$.

19. $x^3 - x^2 + x$.

20. $a^2b + ab^2 + ab^3$. Ans. a, b, and $(a + b + b^2)$.
21. $abc + bcd + acd$. **22.** $x^2y^2 + x^2y + xy^2$.
23. $a^2bc + ab^2c + abc^2$. Ans. a, b, c, and $(a + b + c)$.
24. $ax - bx + cx - dx$. **25.** $3a^4b - 6a^3b^2 + 3a^2b^3$.
26. $8x^2y + 6xy^2 + 2xy$. **27.** $24a^2b - 36ab^2 + 48a^2b^2$.
28. $x(a + b) + y(a + b) + z(a + b)$.
29. $a(2x - y) + b(2x - y) - 3(2x - y)$.
30. $(x + y)^3 - 2(x + y)^2 - a(x + y)$.

57. Type II. A trinomial that is the square of a binomial. (See §§ 49 and 50.)

Exercise 44

Factor the following at sight:

1. $x^2 + 2xy + y^2$.
2. $n^2 - 10n + 25$.
3. $n^2 - 12n + 36$.
4. $4n^2 + 4n + 1$.
5. $4a^2 - 12ab + 9b^2$.
6. $4x^2 - 20x + 25$.
7. $9a^2 - 12ab + 4b^2$.
8. $4m^2 + 16mn + 16n^2$.
9. $25x^2 + 40xy + 16y^2$.
10. $49x^2 + 56xy + 16y^2$.
11. $81 - 36m + 4m^2$.
12. $64a^2 + 32am + 4m^2$.
13. $36x^2 + 84x + 49$.
14. $36x^2 - 108xy + 81y^2$.
15. $a^4 - 2a^2 + 1$.
16. $4a^4 + 12a^2b^2 + 9b^4$.
17. $a^2b^2 + 2ab + 1$.
18. $x^2y^2 + 4xyz + 4z^2$.
19. $81x^2 + 18xyz + y^2z^2$.
20. $(a + b)^2 + 2(a + b) + 1$.
Ans. $(a + b + 1)(a + b + 1)$.
21. $(x - y)^2 + 6(x - y) + 9$.
22. $(a - x)^2 + 12(a - x) + 36$.
23. $4a^2 - 4a(x - y) + (x - y)^2$.
24. $(x + y)^2 - 2(x + y)(a + b) + (a + b)^2$.

58. Type III. A binomial that is the difference of two squares. (See § 51.)

Exercise 45

Factor the following at sight and check by multiplication:

1. $x^2 - 1$. **5.** $4m^2 - 1$. **9.** $1 - 16a^2$.
2. $a^2 - 4$. **6.** $1 - 4m^2$. **10.** $9 - 16b^2$.
3. $b^2 - 9$. **7.** $9a^2 - b^2$. **11.** $16a^2 - 25b^2$.
4. $a^2 - 16$. **8.** $4a^2 - 9b^2$. **12.** $64a^2 - 1$.
13. $81a^2 - 16b^2$. **14.** $36b^2 - 49c^2$. **15.** $9R^2 - 16r^2$.
16. $16a^4 - 1$. Ans. $(4a^2 + 1)$, $(2a + 1)$, and $(2a - 1)$.
17. $81x^4 - 1$. **18.** $16a^4 - 81b^4$. **19.** $81y^4 - 16b^4$.
20. $a^2b^2 - 1$. Ans. $(ab - 1)$ and $(ab + 1)$.
21. $4a^2c^2 - 9$. **22.** $a^2b^2c^2 - 1$. **23.** $16a^4b^4 - 81c^4$.
24. $(a + b)^2 - 1$. Ans. $(a + b + 1)$ and $(a + b - 1)$.
25. $(x - y)^2 - 4$. **26.** $(c + d)^2 - 9$.
27. $(c - d)^2 - a^2$.
28. $a^2 - (b + c)^2$. Ans. $[a + (b + c)]\cdot[a - (b + c)]$, or $(a + b + c)\cdot(a - b - c)$.
29. $m^2 + 2mn + n^2 - 1$. **Hint.** $(m + n)^2 - 1$.
30. $4a^2 + 4ab + b^2 - 9$. **32.** $a^2 - 2ab + b^2 - 81c^2$.
31. $1 - 2x + x^2 - y^2$. **33.** $x^2 + 2xy + y^2 - 64b^4c^2$.
34. $1 - x^2 - 2xy - y^2$. **Hint.** $1 - (x^2 + 2xy + y^2)$. Ans. $(1 + x + y)$ and $(1 - x - y)$.
35. $a^2 - x^2 - 2xy - y^2$.
36. $a^2b^2c^2 - x^2 + 2xy - y^2$.
37. $(a + b)^2 - (c + d)^2$.
38. $(2a - b)^2 - (3c - 4d)^2$.
39. $(x - y - z)^2 - (a - b)^2$.

59. Type IV. A trinomial that is the product of two binomials with a common term. (See § 52.)

Exercise 46

Factor the following at sight and check by multiplication:

1. $n^2 + 5n + 6$.
2. $m^2 + 7m + 12$.
3. $a^2 - 6a + 8$.
4. $b^2 - 7b + 10$.
5. $b^2 - 7b + 6$.
6. $n^2 - 6n + 5$.
7. $x^2 + 8x + 12$.
8. $x^2 + 10x + 9$.
9. $x^2 + 8x + 15$.
10. $x^2 + 8x + 7$.
11. $m^2 - 12m - 13$.
12. $R^2 - 8R - 20$.
13. $R^2 + 12R + 20$.
14. $R^2 - 9R + 20$.
15. $R^2 - R - 20$.
16. $m^2 - mn - 20n^2$.
17. $m^2 - 3mn - 18n^2$.
18. $m^2 - 2mn - 15n^2$.
19. $a^2 - 14ab - 32b^2$.
20. $a^2 - 7ab - 18b^2$.
21. $12 - x - x^2$.

Hint. $12 - x - x^2$
$= -1(x^2 + x - 12)$
$= -1(x + 4)(x - 3)$
$= (x + 4)(3 - x)$.

22. $30 - 13x - x^2$.
23. $20 - 8a - a^2$.
24. $36a^2 - 9ab - b^2$.
25. $18a^2 + 3ab - b^2$.
26. $36R^2 - 5Rr - r^2$.
27. $26x^2 + 11xy - y^2$.

28. Supply the missing numerical coefficient in $x^2 - ?x - 24$ so that the resulting trinomial shall have two binomial factors. In how many ways can this be done?

29. Supply four different Arabic numbers for the missing numerical coefficient of $x^2 - ?x + 24$ so that the resulting trinomial shall have two binomial factors.

30. Supply, in as many ways as possible, the missing numerical coefficient in $a^2 + ?a - 36$ so that the trinomial shall have two binomial factors.

60. Type V. A trinomial that is the product of two binomials with like terms. (See § 53.)

I. Factor $4x^2 - 15xy + 9y^2$.

A trial by multiplication of the various pairs of possible binomial factors will determine the correct pair.

I

$$\begin{array}{r}4x - 9y\\ x - y\\ \hline 4x^2 - 9xy\\ - 4xy + 9y^2\\ \hline 4x^2 - 13xy + 9y^2\end{array}$$

Incorrect

II

$$\begin{array}{r}4x - y\\ x - 9y\\ \hline 4x^2 - xy\\ - 36xy + 9y^2\\ \hline 4x^2 - 37xy + 9y^2\end{array}$$

Incorrect

III

$$\begin{array}{r}2x - 3y\\ 2x - 3y\\ \hline\end{array}$$

See Type II

IV

$$\begin{array}{r}4x - 3y\\ x - 3y\\ \hline 4x^2 - 3xy\\ - 12xy + 9y^2\\ \hline 4x^2 - 15xy + 9y^2\end{array}$$

Correct

It will be observed that it was necessary to try two binomials having the signs of the second terms alike. (Why?) A careful inspection of the trinomial to be factored will enable the student to discover the correct factors with a very few trials.

II. Factor $3a^2 - ab - 10b^2$.

Evidently the signs of the second terms of the binomial factors are unlike.

$$\begin{array}{c}3a + 2b\\ a - 5b\\ \hline 3a^2 - 13ab - 10b^2\end{array}$$

Incorrect

$$\begin{array}{c}3a - 5b\\ a + 2b\\ \hline 3a^2 + ab - 10b^2\end{array}$$

Incorrect

$$\begin{array}{c}3a + 5b\\ a - 2b\\ \hline 3a^2 - ab - 10b^2\end{array}$$

Correct

Exercise 47

Factor the following:

1. $2x^2 + 5x + 3$.
2. $2x^2 - 5x + 3$.
3. $2x^2 - 5x - 3$.
4. $2x^2 + x - 3$.
5. $2x^2 - x - 3$.
6. $2x^2 + 5x - 3$.
7. $3a^2 - 7a + 4$.
8. $3a^2 + 7a + 4$.
9. $3a^2 + 13a + 4$.
10. $3a^2 + 11a - 4$.
11. $3c^2 - 11cd - 4d^2$.
12. $3c^2 - 4cd - 4d^2$.
13. $3a^2 + 10ab + 3b^2$.
14. $3a^2 + 8ab - 3b^2$.
15. $4a^2 + 21ad + 5d^2$.
16. $4a^2 - ad - 5d^2$.
17. $4a^2 + 8ad - 5d^2$.
18. $4a^2 - 7ax + 3x^2$.
19. $5a^2 - 13a + 6$.
20. $5a^2 - 31ad + 6d^2$.
21. $5a^2 - 17ad + 6d^2$.
22. $5a^2 - a - 6$.
23. $5a^2 + ax - 6x^2$.
24. $5a^2 - 7ax - 6x^2$.
25. $5a^2 - 13ay - 6y^2$.
26. $5m^2 + 13mn - 6n^2$.
27. $6a^2 - 11ab - 10b^2$.
28. $6a^2 - 7ab - 10b^2$.
29. Make and factor as many as possible different trinomials by supplying the missing numerical coefficient in $6x^2 - ?xy - 12y^2$.

61. Type VI. The product of two binomials that have unlike terms.

Certain polynomials may be factored by arranging in groups which have a common binomial factor. In the polynomial, $ax + bx - ay - by$, the binomial $ax + bx$ has the factors $x(a + b)$ and the binomial $-ay - by$ has the factors $-y(a + b)$. Therefore $ax + bx - ay - by = x(a + b) - y(a + b) = (x - y)(a + b)$.

Notice that the final factors are obtained by treating the second expression as a binomial that has the monomial factor $(a + b)$.

Exercise 48

Factor the following:

1. $ab + ax + by + xy.$
2. $ab - ax - by + xy.$
3. $2mx + 2nx - my - ny.$
4. $2ax + 2ay + 3bx + 3by.$
5. $3ax - ay - 9bx + 3by.$
6. $6rx + 4sx - 9ry - 6sy.$
7. $cy + 3c + dy + 3d.$
8. $bm - bn - 7m + 7n.$
9. $21 + 3x + 7x^2 + x^3.$
10. $a^2 + 4a + ab + 4b.$
11. $a^2 + ab + ac + bc.$
12. $a^3 + 2a^2 + 3a + 6.$
13. $a^3 + a^2 + a + 1.$
14. $y^2 - xy + yz - xz.$
15. $4ax + 6ay - 10bx - 15by.$
16. $12am - 8an + 3bm - 2bn.$
17. $3a^2m + 3mx^2 - a^2n - nx^2.$
18. $20ar + 8as - 15br - 6bs.$
19. $5a^2x^2 + 5x^2 - 2a^2y^2 - 2y^2.$
20. $8bm + 6cm - 12bn - 9cn.$
21. $m^2x^2 - m^2y^2 + n^2x^2 - n^2y^2.$ (Find three factors.)
22. $2a^2m - 2b^2m + 3a^2n - 3b^2n.$
23. $2r^2x - 2s^2x + 3r^2y - 3s^2y.$

The process used in Type VI may be made to assist in the solution of the problems of Type V in the following manner:

Since
$$\begin{aligned}(3a - 2b)(4a + 5b) &= 3\cdot4a^2 + 3\cdot5ab - 2\cdot4ab - 2\cdot5b^2\\ &= 12a^2 + 15ab - 8ab - 10b^2\\ &= 12a^2 + 7ab - 10b^2,\end{aligned}$$
it is evident that $+7$, the numerical coefficient of the middle term, is the sum of two factors of -120, which is the product of the numerical coefficients of the first and last terms. -120 is the product of 3, -2, 4, and 5 while $+7$ is the sum of $3\cdot5$ and $-2\cdot4$.

Therefore, in factoring $12a^2 + 7ab - 10b^2$, take the product of 12 and -10 which is -120 and examine all its pairs of factors beginning with $+1$ and -120, -1 and $+120$, etc., until a pair is found whose sum is $+7$. This pair is $+15$ and -8.

Then $12a^2 + 7ab - 10b^2 = 12a^2 + 15ab - 8ab - 10b^2$

$$= 3a(4a + 5b) - 2b(4a + 5b)$$

$$= (3a - 2b)(4a + 5b).$$

Similarly, in factoring $15x^2 - 11x - 12$, $15(-12)$ gives -180. Rejecting all pairs of factors until 9 and -20 are reached, we have $15x^2 - 11x - 12 = 15x^2 - 20x + 9x - 12$

$$= 5x(3x - 4) + 3(3x - 4)$$

$$= (5x + 3)(3x - 4).$$

Exercise 49

Factor the following:

1. $3n^2 + 5n + 2$.
2. $3n^2 - 5n + 2$.
3. $5r^2 + 7r + 2$.
4. $5r^2 - 7r + 2$.
5. $6m^2 - 13m + 6$.
6. $6x^2 + 13xy + 6y^2$.
7. $6a^2 - 5a - 6$.
8. $7a^2 - a - 8$.
9. $6r^2 - 13r - 5$.
10. $6n^2 - 19n - 7$.
11. $4n^2 + 8n - 5$.
12. $5s^2 - 31s + 6$.
13. $2n^2 - n - 10$.
14. $2a^2 - 3ab - 9b^2$.
15. $6a^2 - 11ax - 7x^2$.
16. $6a^2 - 11a - 10$.
17. $8x^2 + 6x - 9$.
18. $10m^2 - 31m - 14$.
19. $14a^2 - 33ab + 18b^2$.
20. $8d^2 + 18cd + 9c^2$.
21. $14m^2 + 73mn + 15n^2$.
22. $9h^2 + 21hd + 10d^2$.
23. $9x^2 - 4xy - 13y^2$.
24. $18a^2 - 9ab - 20b^2$.
25. $10a^2 - 24ab + 14b^2$.
26. $10a^2 - 72ab + 14b^2$.
27. $9a^2 - 33ab + 10b^2$.
28. $18x^2 + 31xy - 20y^2$.
29. $12b^2 - 20bc - 25c^2$.
30. $12a^2 - 32ac - 12c^2$.
31. $25x^2 + 130xy + 25y^2$.
32. $24x^2 - 50xy + 24y^2$.
33. $24a^2 + 25ay - 25y^2$.
34. $20b^2 - 44bc - 27c^2$.

62. Suggestions for factoring. The greatest difficulty the student will encounter in factoring will be that of recognizing under which type the given expression is to be factored. This text contains in its first ten chapters no exercises in factoring that do not come under one or more of the preceding six types.

Study carefully the following directions:

(1) Examine the expression to determine if it contains a monomial factor.

(2) If the remaining factor is a binomial, see if **Type III** applies; if it is a trinomial, test it for **Types II, IV,** and **V;** if it is a polynomial of four terms, it may be the difference of two squares as $a^2 + 2ab + b^2 - c^2 = (a + b)^2 - c^2$, or it may belong to **Type VI.**

(3) It must be remembered that the expression is not completely factored until all its prime factors are found.

(4) Prove the accuracy of your work by multiplying together the resulting factors to see whether their product is the original expression.

Illustrative examples.

1. $x^3 - 4x^2 + 4x = x(x^2 - 4x + 4) = x(x - 2)(x - 2).$

2. $a^3 - 5a^2 + 4a = a(a^2 - 5a + 4) = a(a - 4)(a - 1).$

3.
$$\begin{aligned} m^2 - 1 - 2mn + n^2 &= m^2 - 2mn + n^2 - 1 \\ &= (m - n)^2 - 1 \\ &= (m - n + 1)(m - n - 1). \end{aligned}$$

4. $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) =$
$$(x^2 + y^2)(x + y)(x - y).$$

Note. The student may write the results, as in No. 1 above, using the = sign to show that the original expression equals the product of the factors, or he may write, "the factors of $x^3 - 4x^2 + 4x$ are x, $(x - 2)$, and $(x - 2)$."

Exercise 50

Reduce the following to prime factors:

1. $12x^3y^2$.
2. $36a^2b^3c^4$.
3. $2a^3 + 4a^2b$.
4. $x^2 - 4x$.
5. $9r^2 - 12rt + 4t^2$.
6. $a^2c + 2abc + b^2c$.
7. $25 - x^2$.
8. $50x - 18x^3$.
9. $b^2 + 9b - 22$.
10. $a^3 - 20a^2 - 44a$.
11. $7a^2 - 15ab + 2b^2$.
12. $14x^2 - 21x + 7$.
13. $xy - xz - yz + z^2$.
14. $3m^2 + 12m + 3mn + 12n$.
15. $49a^2 - 121b^2$.
16. $98m^2 - 162$.
17. $x^3 - 100x$.
18. $m^4 - r^4$.
19. $4n^4 + 5n^2 - 9$.
20. $a^4b^3 - a^2b^5$.
21. $6mx + 9nx - 4my - 6ny$.
22. $25n^5 + 70n^4 + 49n^3$.
23. $x^4 - 5x^2 + 4$.
24. $2a^2 + 33ax + 31x^2$.
25. $48x^2 - 23xz - 13z^2$.
26. $c^2 - 2cd + d^2 - k^2$.
27. $9n^2 - 30n + 25 - r^2$.
28. $36a^2 - 84ab + 49b^2 - 64c^2$.
29. $25 - (c + d)^2$. Ans. $5 - c - d$ and $5 + c + d$.
30. $36 - (x - y)^2$.
31. $49x^2 - (3y - 5z)^2$.
32. $x^2 - m^2 - 2mn - n^2$.
33. $25 - a^2 + 2ab - b^2$.
34. $36r^2 - 9x^2 + 42xy - 49y^2$.
35. $(x + y)^2 - (a + b)^2$.
36. $(x - y)^2 - 4(x - y) + 4$. Ans. $x - y - 2$ and $x - y - 2$.
37. $(m - n)^2 - 12(m - n) + 36$.
38. $(x + y)^2 + 3(x + y) - 18$.
39. $(a + b)^2 - (a + b) - 20$.

40. $10 - 3(m + n) - (m + n)^2$.

41. $y^4 + 26y^2 - 27$.

42. $1 - 12a - 28a^2$.

43. $n^6 - 9n^3 + 20$.

44. $7a^2b^4 - 3ab^2 - 4$.

45. $9m^2n^2 + 30mn + 25$.

46. $5x + 4x^2 - x^3$.

47. $100R^2 - 121r^2$.

48. $3a^2b^3 + 12a^2b^2 + 12a^2b$.

49. $10rx - 6ns - 4nr + 15sx$.

50. $2ax^2 - 3ny^2 + 3nx^2 - 2ay^2$.

51. $x^2 - 2xy + y^2 - a^2 - 2ab - b^2$.

52. $(a - b)^2 - (a - b)(x - y) - 2(x - y)^2$.

53. $(m - n - 2)^2 - 2(m - n - 2)a + a^2$.

54. $(a - b)^3 - 2(a - b)^2 + (a - b)$.

QUADRATIC EQUATIONS

63. Definitions. The **degree of a term** is determined by the number of literal factors that it contains, provided that none of those factors are in the denominator.

a, $3a$, and $6c$ are terms of the **first degree**; ab, a^2, and $3xy$ are terms of the **second degree**; and abc, a^2b, and $2mn^2$ are terms of the **third degree.**

If a certain literal number does not appear in the denominator, the degree of a term with respect to that number is determined by its exponent.

$2a^2x^3y$ is of the second degree with respect to a, of the third degree with respect to x, and of the first degree with respect to y.

Find the degree of $4ab^2c^3x^4$ with respect to each literal number.

The **degree of a polynomial** is determined by its term of highest degree. The polynomial $ax^2 + bx + c$ is of the third degree since its term of highest degree, ax^2, is of the third degree. It is also of the second degree with respect to x. Why?

A **quadratic equation** in a certain literal number is an equation of the second degree with respect to that number, provided that the literal number does not appear in any term in the denominator.

$ax^2 + bx + c = 0$ is a quadratic equation with respect to, or in x.

A quadratic equation is frequently called an **equation of the second degree.**

$7b^2 + 3b - 4 = 0$ is a quadratic equation in b.

64. Solution of quadratic equations by factoring. Many quadratic equations may be solved by factoring, provided we assume the following as an axiom:

Axiom V. *If the product of two or more factors is zero, one of the factors must be zero.*

In arithmetic we have learned that if one factor is equal to zero, the product is zero. Of course two or more or even all the factors may be equal to zero and the product will be zero.

The quadratic equation $x^2 - 3x + 2 = 0$ may be put by factoring in the form $(x-2)(x-1) = 0$.

This will be an equation if either $x - 2$ or $x - 1$ is equal to zero.

If $x - 1 = 0$, then $x = 1$. (What axiom is used here?)

If $x - 2 = 0$, then $x = 2$. (Why?)

Therefore in the equation $x^2 - 3x + 2 = 0$, x may equal 1 or 2. Checking by substituting 1 for x, we get $1 - 3 + 2 = 0$, or $0 = 0$. Therefore $x = 1$. In like manner, check for $x = 2$.

Attempt to check the above equation by substituting other numbers for x, such as 3, 4, 5, and 10.

Apparently $x^2 - 3x + 2 = 0$ is true only when $x = 1$, or $x = 2$. These values are called the **roots** of the equation.

Exercise 51

Find the roots of the following:

1. $(x-2)(x-3)=0.$
2. $(x+2)(x-3)=0.$
   Ans. -2 and 3.
3. $(2x-1)(x+4)=0.$
4. $(x-1)(x-3)=0.$
5. $(2x+3)(2x-3)=0.$

Solve the following by factoring and check:

6. $x^2-4x+3=0.$
7. $x^2-5x+6=0.$
8. $x^2-7x+6=0.$
9. $x^2-8x+16=0.$
10. $x^2-4x-12=0.$ **Hint.** $(x-6)(x+2)=0.$
11. $x^2-9x-36=0.$
12. $x^2-2x=-1.$ **Hint.** $x^2-2x+1=0.$ (Why?)
13. $x^2-2x-1=7.$
14. $x^2=-4x.$ Ans. $x=0$, and $x=-4.$
15. $x^2-x=0.$
16. $x^3-x=0.$
17. $x^2-2x-1=14.$
18. $n^2-10=3n.$
19. $a^2-4=0.$
20. $4x^2-1=0.$
21. $a^3-4a^2+4a=0.$
22. $2n^2-5n+3=0.$
23. $(x-2)^2+(x-1)(x-3)=71.$

Suggestion. Expand, collect all terms in the left member, and solve by factoring.

24. $(x-2)^2+(x-3)^2=25.$
25. $(n-2)(n-3)+(n+2)=5.$
26. Write the equation whose roots are 3 and 4.

Suggestion. Show that the equation must be $(x-3)(x-4)=0$, which becomes $x^2-7x+12=0.$

Write the quadratic equations having the following roots:

27. 1 and 5.
28. -2 and 4.
29. -3 and $-2.$
30. 3 and $-5.$

Exercise 52. Problems Involving Quadratics

1. The square of a certain number plus the number itself equals 6. What is the number? Ans. 2 or -3.

2. Find the number whose square added to itself gives 20.

3. The length of a city lot is 1 rod less than twice its width and it contains 28 square rods. What are its dimensions in rods?

Suggestion. $x(2x-1)=28$. Whence $2x^2-x-28=0$. Or $(2x+7)(x-4)=0$. Ans. $x=4$ or $x=-3\frac{1}{2}$.

The second answer of the above equation does not satisfy the conditions of the problem since the width of a lot cannot be a negative number of rods. We must keep in mind that we are solving the problem and any answers that do not satisfy the conditions of the problem must be dropped even though they may be true answers of the equation used to solve the problem. The answer $x=4$ does satisfy the conditions.

4. A garden plot in the shape of a rectangle is 3 feet longer than it is wide and contains 40 square feet. Find its dimensions. Ans. 5 feet by 8 feet.

5. Find two consecutive numbers whose product is 42. Ans. 6 and 7, or -7 and -6.

6. What number must be added to both 4 and 8 so that the product of the results shall be 77?

Suggestion. $(x+4)(x+8)=77$.

7. The area of the floor of a certain room whose length is 3 yards more than its width is 54 square yards. Find its dimensions in yards.

8. The sum of the squares of two consecutive numbers is 41. Find the numbers. Ans. 4 and 5, or -5 and -4.

9. The sum of the squares of what two consecutive odd numbers is 74?

10. The sum of the squares of what two consecutive even numbers is 100?

11. A sheet of paper is 3 inches longer than wide and its area is 88 square inches. Find its dimensions.

12. The difference between two numbers is 6 and the sum of their squares is 146. Find the numbers.

13. The side of one square is 5 feet longer than the side of another square. Find the dimensions of the two squares if the sum of their areas is 157 square feet.

14. The sum of the squares of what three consecutive numbers is 194?

15. The sum of the squares of what three consecutive even numbers is 200?

16. What number must be subtracted from both 11 and 9 so that the product of the remainders shall be 63?

17. The sum of the areas of two squares is 74 square feet and the sum of their perimeters is 48 feet. Find their respective sides.

Hint. Let $x =$ the number of feet in the side of one square. Then $4x$ and $48 - 4x$ are the number of feet in the perimeters of the two squares. One fourth of $48 - 4x$ or $12 - x$ is the length in feet of a side of the second square.

18. The sum of the areas of two squares is 169 square feet and the difference of their perimeters is 28 feet. Find their respective sides.

HIGHEST COMMON FACTOR

65. The **highest common factor** (H. C. F.) of two or more given algebraic expressions is the algebraic expression with the largest numerical coefficient and of highest degree that is a common factor of the given expressions.

The H. C. F. is found as in arithmetic by factoring each expression completely and then taking the product of all the common factors.

To find the H. C. F. of $3a^2b$, $6a^2b^2$, and $12xab^3$, we factor the numbers:

$3a^2b = 3 \cdot a \cdot a \cdot b$, $6a^2b^2 = 2 \cdot 3 \cdot a \cdot a \cdot b \cdot b$, and
$12xab^3 = 2 \cdot 2 \cdot 3 \cdot x \cdot a \cdot b \cdot b \cdot b$.

The factors common to all three expressions are 3, a, and b, therefore $3 \cdot a \cdot b$, or $3ab$, is the H. C. F.

Similarly, the H. C. F. of several polynomials such as $a^3 - ab^2$, $2a^3 - 2a^2b$, and $a^3 - 2a^2b + ab^2$ is found, as follows:

$a^3 - ab^2 = a(a - b)(a + b)$.
$2a^3 - 2a^2b = 2 \cdot a \cdot a(a - b)$.
$a^3 - 2a^2b + ab^2 = a(a - b)(a - b)$.

The common factors are a and $a - b$. Therefore $a(a - b)$ is the H. C. F.

Exercise 53

Find the H. C. F. of each of the following groups:

1. $6x^2y^2$, $12x^3y^2z$, $18x^4yz$, and $24x^3y^3$.
2. $25abcd^2$, $30ab^2cd$, $45a^2b^2c$, and $50abc^2$.
3. $a^2 - 4$, $a^2 - a - 2$, and $a^2 - 4a + 4$.
4. $a^3 - a^2b$, $a^3 - 2a^2b + ab^2$, and $a^4 - a^2b^2$.
5. $a^2 - 2a - 8$, $a^2 + 4a - 32$, $a^2 - 16$, and $a^2 - 8a + 16$.
6. $a^2 + 3a - 10$, $a^2 + 6a + 5$, and $a^2 + 2a - 15$.
7. $2x^2y + xy - 6y$, $x^2y + 3xy + 2y$, and $x^2y - 4y$.
8. $a^3 + a^2b$, $a^4 + 2a^3b + a^2b^2$, and $a^4 - a^2b^2$.

LOWEST COMMON MULTIPLE

66. A **multiple** of a given number is any number of which the given number is a factor. The **lowest common multiple** (L. C. M.) of two or more given algebraic expressions is the expression with the least numerical coefficient and of lowest degree that will contain each of the given expressions as a factor.

If one number is divisible by another number, the first number must have all the factors of the second number. That is, any number that is divisible by 12 must have the factors of 12, or 2, 2, and 3. Then, in order to find the lowest common multiple of two or more given numbers such as 12, 20, and 18, we find the factors of the numbers and take each factor as many times as it occurs in any one of the numbers. Therefore we use the factor 2 twice, the factor 3 twice, and the factor 5 once. The L. C. M. is $2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$, or 180.

The L. C. M. of $50ab^2c$, $25a^2bx^3$, and $75c^2x$ is $150a^2b^2c^2x^3$. Explain.

Exercise 54

Find the L. C. M. of each of the following groups:

1. a^2bx, ab^2x, abx^2, and abx.
2. $3ab$, $6bc$, $8abc$, $12abcd$, and $5a^2b^2$.
3. ab, cd, ef, fb, bc, de and fa.
4. a^2b^3c, $a^3b^4c^2$, $a^4b^3c^2$, and ab^2c^3.
5. $a^2 - b^2$, $a^2 - 2ab + b^2$, and $a - b$.
6. $a(x - y)$, $b(x + y)$, $x^2 - y^2$, and $(x - y)^2$.
7. $x^2 - x - 2$, $x^2 + 2x - 8$, $x^2 - 4$, and $x^2 - 4x + 4$.
8. $x^2 + 2xy + y^2 - 9$, $(x + y - 3)^2$, and $3(x + y - 3)$.
9. $n^2 - n$, $n^3 - n$, $n^2 - 2n + 1$, and $n^2 - 3n + 2$.

Exercise 55. General Review

1. Define factor, common factor, coefficient, exponent, term, like terms, degree of a term, degree of an expression, binomial, polynomial, trinomial, equation, equation of condition, identity, root, and absolute or numerical value of a number.

2. State the rules of signs for addition, subtraction, multiplication, and division.

3. What is the degree of $3a^2b^3c^4x$ with respect to a? with respect to b? to c? to x?

4. In $3a^2b^3c^4x$ what is the coefficient of a? of b^2? of abc?

5. State the four axioms which may be used in solving an equation.

6. Arrange in descending powers with respect to a the following:

$2a^4 + 3a^2x^2 - 5a^3x + 7x^4 + 2ax^3.$

What can you say of the result with respect to x?

7. Expand at sight the following, stating each time the formula used:

$(2 + 3x)(2 + 3x)$, $(3y - a)(3y + a)$, $(n - 5)(n + 3)$, $(2 - 3x)(3 + 2x)$, $(ax - 4)(ax + 3)$, $(2 + 3x)(2 - 3x)$.

8. Supply the missing term in each of the following trinomial squares:

$9x^2 + 12x + ?$ $16a^2 + 24ab + ?$ $100n^2 - 20n + ?$
$16h^2 - ? + 9x^2$, $49 - ? + 36a^2b^4c^4$, $? + 20an + n^2$.

9. Factor each of the following:

$16 - 9x^2$; $x^2 - 3x - 28$; $6x^2 - x - 2$; $a^3x^2 - a^2x^2 - 2ax^2$; $6a^2 + 5am - 6m^2$; $10r^2 + 29rs - 21s^2$; $12x^2 - 7x - 12$; $x^4 - y^4$; $x^2 - ax - bx + ab$; $m^2 - 2mn + n^2 - 1$.

10. Find the value, or the values, of the literal numbers in each of the following equations and check:

(a). $4 + 5(x + 2) + 9x = (x + 2)^2 - x^2$.

(b). $(x - 1)(x + 3) - 2(x + 1)(x - 5) + x^2 = 0$.

(c). $2x^2 - 7x = 15$.

(d). $(2x + 3)(x - 5) = (x - 5)(x + 3)$.

11. From $-3a^2 + 8a + 36$ take $a^2 - 2a - 18$.

12. Simplify $6y - (4y + 3z) - (2y - 4z) + 2(y - 2z)$.

13. Multiply $12y^2 - 6y + 2$ by $y^2 - 3y - 1$.

14. Divide $8a^3 - 12a^2 + 6a - 1$ by $2a - 1$.

15. Place within a sign of aggregation preceded by a $-$ sign the last three terms of $a^3 - 3a^2b + 3ab^2 - b^3$. The last two, $a^2 - 2ab + b^2$. The whole expression $a^2 + 2ab + b^2$.

Check in each of the following:

16. Multiply $x^2 + y^2 + z^2 - xy - xz - yz$ by $x + y + z$.

17. Multiply $a^2 + 2ab + b^2 + a + b + 1$ by $a + b - 1$.

18. Multiply $1 - 2a + 2a^2 - a^3$ by $1 + 2a + 2a^2 + a^3$.

19. Divide $a^6 - 2a^3 + 1$ by $a^3 - 1$.

20. Divide $16x^4 - 1$ by $2x - 1$.

21. Divide $a^5 + b^5$ by $a + b$.

22. Divide $a^5 - b^5$ by $a - b$.

23. Solve for a: $22 - 5(3 - 2a) = a - 4(a + 8)$.

24. Solve for b: $7(3 - b) - 4(7 - 2b) = 0$.

25. Solve for m: $(m + 3)(m + 5) = m^2 + 31$.

26. Solve and check: $7w^2 - 15w + 2 = 0$.

27. Solve and check: $t^2 - t = 72$.

28. Find four consecutive numbers whose sum is 66.

29. Find two consecutive numbers whose product is 72.

30. Find three consecutive numbers the sum of whose squares is 29. Ans. 2, 3, and 4; and -4, -3 and -2.

31. The Ohio River is 110 miles more than 3 times as long as the Hudson and the Colorado is 40 miles less than 5 times as long as the Hudson. If the length of the Colorado is subtracted from twice the length of the Ohio, the remainder is 440 miles. Find the length of each of the three rivers.

32. The second of two numbers is 2 more than 3 times the first and the third is 4 less than 5 times the first. If the third is subtracted from the sum of the first and second, the remainder is 1. Find the numbers.

33. Separate 15 into two parts such that their product shall be 56.

34. One number is 4 more than another and the sum of their squares is 58. What are the numbers?

35. I have $3.75 in dimes and quarters. How many of each kind of coin have I, if I have 18 coins altogether?

36. The difference of two numbers is 2 and the difference of their squares is 16. What are the numbers?

37. The perimeter of a certain rectangle is 28 inches and its area is 48 square inches. Find its dimensions.

38. The sum of two consecutive numbers is 5 less than their product. Find the numbers.

39. James walks 15 miles. The number of hours he walks is 2 more than the number of miles he walks per hour. Find both numbers.

40. The product of two consecutive numbers is 31 less than the sum of their squares. What are the numbers?

CHAPTER V

FRACTIONS

67. Definitions. We have learned (§ 8) that the quotient of two literal numbers, such as a divided by b, may be written in the form $\frac{a}{b}$, which is called a fraction, and may be read "the fraction a divided by b", or "the fraction a over b," or simply "a over b."

In algebra, as in arithmetic, the **dividend is called** the **numerator,** the **divisor** the **denominator,** both together the **terms of the fraction,** and the quotient the **value of the fraction.**

In arithmetic the **value of the fraction** can be expressed as a decimal. This is impossible in the case of a literal fraction since the numerical values of its terms are usually not known.

An **integer,** or an **integral number,** is a number that is not fractional.

68. Reduction of a fraction to lower terms. In algebra as in arithmetic it is frequently possible to reduce a given fraction to an equivalent fraction whose terms are **simpler** or of **lower degree.** This process rests on the assumption of the following axiom:

Axiom VI. *If both terms of a fraction are multiplied or divided by the same number, the value of the fraction is not changed.*

Rule. *To simplify an algebraic fraction, divide both terms by their H. C. F.*

Exercise 56

Simplify the following:

1. $\frac{4}{8}$. 2. $\frac{10}{12}$. 3. $\frac{15}{25}$. 4. $\frac{45}{65}$. 5. $\frac{26}{91}$.

6. $\frac{57}{95}$. 7. $\frac{6a}{8a} = \frac{3 \cdot 2a}{4 \cdot 2a} = \frac{3}{4}$. 8. $\frac{2ab}{5ab}$. 9. $\frac{3ab}{6ac}$.

10. $\frac{7xy}{14xyz}$. 11. $\frac{3ab}{5axy}$. 12. $\frac{8abc}{12acx}$. 13. $\frac{10xy}{15wxyz}$.

14. $\frac{12axz}{32ayz}$. 15. $\frac{4a}{6a^2} = \frac{2 \cdot 2a}{3a \cdot 2a} = \frac{2}{3a}$. 16. $\frac{12x}{16x^2y}$.

17. $\frac{12ab}{16a^2b}$. 18. $\frac{8ab}{14a^2bc}$. 19. $\frac{15x^2}{25x^3}$. 20. $\frac{12ax^2}{16a^2x}$.

21. $\frac{10axy}{12a^2xy^2}$. 22. $\frac{10a^2b}{15a^2b^2c}$. 23. $\frac{12abm^2}{32a^2m}$.

24. $\frac{11ab^2c^3}{33a^2b^2c}$. 25. $\frac{10a^2bc}{15bx}$. 26. $\frac{18xy^2z}{48x^2yz^2}$.

27. $\frac{3a}{3a - 6b} = \frac{3a}{3(a - 2b)} = \frac{a}{a - 2b}$.

28. $\frac{m}{m^2 + m}$. Ans. $\frac{1}{m + 1}$. 29. $\frac{ab}{ax + ay}$.

30. $\frac{2ax}{2a^2 + 4ay}$. 31. $\frac{2a^2xy}{4ay - 6ay^2}$. 32. $\frac{2x^2y^3}{x^2y + xy^2}$.

33. $\frac{ab - bc}{ab + bc} = \frac{b(a - c)}{b(a + c)} = \frac{a - c}{a + c}$.

34. $\frac{4 - 2a}{6 + 8a}$. 35. $\frac{3a^2 - ab}{2a^2b + 3ab^2}$. 36. $\frac{a^2b - ab^2}{a^3b^2 + a^2b^3}$.

37. $\frac{ax - 3a}{x^2 - 9} = \frac{a(x - 3)}{(x + 3)(x - 3)} = \frac{a}{x + 3}$.

38. $\dfrac{x^2 - a^2}{ax - a^2}$. Ans. $\dfrac{x + a}{a}$. **39.** $\dfrac{x^2 - a^2}{x^2 + 2ax + a^2}$.

40. $\dfrac{m^2 - m - 2}{m^2 - 4m + 4}$. **41.** $\dfrac{am^2 - 4a}{m^2 + 4m + 4}$. **42.** $\dfrac{x^2 - 2ax + a^2}{x^2 + ax - 2a^2}$

43. $\dfrac{a(x^2 - 4y^2)}{ax^2 - 5axy + 6ay^2}$. **44.** $\dfrac{3a^2x - 12x^3}{2a^2 + 8ax + 8x^2}$.

69. The three signs of a fraction. Since a fraction is an indicated division, with dividend, divisor, and quotient becoming the numerator, denominator, and value of the fraction, three signs must be written or understood.

These are, (1) the sign of the numerator (or dividend), (2) the sign of the denominator (or divisor), and (3) the sign of the value of the fraction (or quotient), written before the fraction and on the line of its bar.

The value of the fraction $\dfrac{12}{3}$ is $+4$; therefore $-\dfrac{12}{3}$ is -4. ($-\dfrac{12}{3}$ is to be read "the negative fraction 12 divided by 3".)

Similarly, $\dfrac{-12}{3} = -4$. (See § 31.) $\dfrac{12}{-3} = -4$, also $-\dfrac{-12}{-3} = -4$.

Evidently, $\dfrac{-12}{3} = \dfrac{12}{-3} = -\dfrac{12}{3} = -\dfrac{-12}{-3}$.

Can you explain why $\dfrac{16}{-2} = \dfrac{-16}{2} = -\dfrac{16}{2} = -\dfrac{-16}{-2}$?

Using the numbers 16 and 2, can you write four fractions whose values are all $+8$?

Write three other fractions, using a and b, each one equal to $-\dfrac{a}{b}$.

Write three other fractions each equal to $\dfrac{-a}{-b}$.

The **rule** of § 31 in terms of fractions is,—

If the numerator (dividend) and denominator (divisor) have like signs, the value of the fraction (quotient) is positive. If they have unlike signs, the value of the fraction is negative.

That is $\frac{+a}{+b} = \frac{-a}{-b}$ and $\frac{-a}{+b} = \frac{+a}{-b}$.

This gives the following:

Rule. *If the signs of both terms of a fraction are changed, the sign of the value of the fraction is not changed; but if the sign of one term is changed, the sign of the value of the fraction is changed.*

Exercise 57

Reduce each of the following fractions to its lowest terms, expressing the value of the fraction as negative if necessary:

1. $\frac{-6}{-8}$. Ans. $\frac{3}{4}$.

2. $\frac{-a}{b}$. Ans. $-\frac{a}{b}$.

3. $\frac{6a}{-8a^2} = -\frac{2a\cdot 3}{2a\cdot 4a} = -\frac{3}{4a}$.

4. $\frac{7xy}{-14x^2}$.

5. $\frac{10abc}{-35bcd}$.

6. $\frac{-8a^2b}{-12ab^2}$

7. $\frac{-a(a+b)}{-(a+b)(a+b)}$.

8. $\frac{ax-bx}{a^2-2ab+b^2}$.

9. $\frac{x^2-y^2}{x^2-2xy+y^2}$.

10. $\frac{x^2-y^2}{x^2+2xy+y^2}$.

11. $\frac{a(m-n)}{n^2-m^2} = \frac{a(m-n)}{-(m^2-n^2)} = -\frac{a}{m+n}$.

12. $\dfrac{a(x-y)}{y^2-x^2}$. 13. $\dfrac{(2x-1)a}{(1-4x^2)b}$.

14. $\dfrac{a-b}{(b-a)(b+c)}$. Ans. $-\dfrac{1}{b+c}$.

15. $\dfrac{a+1}{a^2+4a+3}$. 16. $\dfrac{x^2+2x+1}{x^2+3x+2}$. 17. $\dfrac{a^2-6a+9}{a^2-a-6}$.

18. $\dfrac{m^2+3m-10}{m^2+6m+5}$. 19. $\dfrac{a^2-3a}{a^2-6a+9}$

20. $\dfrac{a^2+2ab+b^2-x^2}{a^2-(b+x)^2}$. 21. $\dfrac{5-x}{x^2-7x+10}$.

22. $\dfrac{a^2-10ab+16b^2}{a^2-16ab+64b^2}$. 23. $\dfrac{a^2+8ab+12b^2}{a^2+9ab+18b^2}$.

24. $\dfrac{(x-y)^2-z^2}{x^2-(y+z)^2}$. 25. $\dfrac{2a^2-5a+3}{5a^2-12a+7}$.

70. Fractions changed to equivalent fractions with a given denominator or numerator.

Change $\dfrac{a}{2b}$ to an equivalent fraction whose denominator is $8b$.

It is evidently necessary to supply the missing term in $\dfrac{a}{2b}=\dfrac{?}{8b}$. $2b\cdot 4=8b$, therefore both terms of $\dfrac{a}{2b}$ must be multiplied by 4. (What axiom?) Therefore $\dfrac{a}{2b}=\dfrac{4a}{8b}$.

Exercise 58

Supply the missing terms in each of the following:

1. $\dfrac{2}{3}=\dfrac{?}{24}$. 2. $\dfrac{2x}{3y}=\dfrac{?}{9y}$. 3. $\dfrac{2x}{3y}=\dfrac{?}{12xy}$.

4. $\dfrac{ax}{by}=\dfrac{?}{by\,(m-n)}$. 5. $\dfrac{3a}{4x}=\dfrac{?}{4bx\,(a-b)}$.

6. $\dfrac{a}{a+b}=\dfrac{?}{b^2-a^2}$. 7. $\dfrac{a}{a-b}=\dfrac{?}{a^2-b^2}$.

8. $\dfrac{a}{a-b}=\dfrac{?}{b^2-a^2}$. 9. $\dfrac{a}{(a-b)}=\dfrac{?}{(b-a)\,(b-c)}$.

10. $\dfrac{5x}{5+x}=\dfrac{?}{25-x^2}$. 11. $\dfrac{1}{5-x}=\dfrac{?}{x^2-25}$.

12. $\dfrac{1}{a(x-1)}=\dfrac{?}{a^2(1-x^2)}$.

13. $\dfrac{x}{x+y+z}=\dfrac{?}{x^2-(y+z)^2}$.

14. $\dfrac{a+b}{a-b}=\dfrac{?}{a^2-b^2}$. 15. $\dfrac{a-3}{a+3}=\dfrac{a^2-9}{?}$.

16. $\dfrac{x-y}{x}=\dfrac{x^2+xy-2y^2}{?}$. 17. $\dfrac{ab}{a+2b}=\dfrac{a^2b+ab^2}{?}$

18. $\dfrac{2}{c^2+d^2}=\dfrac{2c^2-2d^2}{?}$. 19. $\dfrac{a+1}{a+2}=\dfrac{a^2+4a+3}{?}$.

20. $\dfrac{a-b-c}{a+c-b}=\dfrac{a^2-(b+c)^2}{?}$

71. Mixed expressions. A **mixed expression** is an expression consisting of two parts, one integral, the other fractional.

$2x-\dfrac{3}{ax}$ and $a^2+a-\dfrac{8}{a}$ are mixed expressions.

In arithmetic certain fractions can be reduced to an equivalent integer or a mixed expression by performing the indicated division. Such fractions are called **improper fractions.**

$\dfrac{12}{4}$ and $\dfrac{15}{4}$ are improper fractions. The first equals 3 and the second equals $3\frac{3}{4}$ when the numerator is divided by the denominator.

12. $\dfrac{a(x-y)}{y^2-x^2}$.

13. $\dfrac{(2x-1)a}{(1-4x^2)b}$.

14. $\dfrac{a-b}{(b-a)(b+c)}$. Ans. $-\dfrac{1}{b+c}$.

15. $\dfrac{a+1}{a^2+4a+3}$.

16. $\dfrac{x^2+2x+1}{x^2+3x+2}$.

17. $\dfrac{a^2-6a+9}{a^2-a-6}$.

18. $\dfrac{m^2+3m-10}{m^2+6m+5}$.

19. $\dfrac{a^2-3a}{a^2-6a+9}$

20. $\dfrac{a^2+2ab+b^2-x^2}{a^2-(b+x)^2}$.

21. $\dfrac{5-x}{x^2-7x+10}$.

22. $\dfrac{a^2-10ab+16b^2}{a^2-16ab+64b^2}$.

23. $\dfrac{a^2+8ab+12b^2}{a^2+9ab+18b^2}$.

24. $\dfrac{(x-y)^2-z^2}{x^2-(y+z)^2}$.

25. $\dfrac{2a^2-5a+3}{5a^2-12a+7}$.

70. Fractions changed to equivalent fractions with a given denominator or numerator.

Change $\dfrac{a}{2b}$ to an equivalent fraction whose denominator is $8b$.

It is evidently necessary to supply the missing term in $\dfrac{a}{2b}=\dfrac{?}{8b}$. $2b\cdot 4=8b$, therefore both terms of $\dfrac{a}{2b}$ must be multiplied by 4. (What axiom?) Therefore $\dfrac{a}{2b}=\dfrac{4a}{8b}$.

Exercise 58

Supply the missing terms in each of the following:

1. $\dfrac{2}{3}=\dfrac{?}{24}$.

2. $\dfrac{2x}{3y}=\dfrac{?}{9y}$.

3. $\dfrac{2x}{3y}=\dfrac{?}{12xy}$.

4. $\dfrac{ax}{by}=\dfrac{?}{by(m-n)}$.

5. $\dfrac{3a}{4x}=\dfrac{?}{4bx(a-b)}$.

6. $\dfrac{a}{a+b}=\dfrac{?}{b^2-a^2}$.

7. $\dfrac{a}{a-b}=\dfrac{?}{a^2-b^2}$.

8. $\dfrac{a}{a-b}=\dfrac{?}{b^2-a^2}$.

9. $\dfrac{a}{(a-b)}=\dfrac{?}{(b-a)\,(b-c)}$.

10. $\dfrac{5x}{5+x}=\dfrac{?}{25-x^2}$.

11. $\dfrac{1}{5-x}=\dfrac{?}{x^2-25}$.

12. $\dfrac{1}{a(x-1)}=\dfrac{?}{a^2(1-x^2)}$.

13. $\dfrac{x}{x+y+z}=\dfrac{?}{x^2-(y+z)^2}$.

14. $\dfrac{a+b}{a-b}=\dfrac{?}{a^2-b^2}$.

15. $\dfrac{a-3}{a+3}=\dfrac{a^2-9}{?}$.

16. $\dfrac{x-y}{x}=\dfrac{x^2+xy-2y^2}{?}$.

17. $\dfrac{ab}{a+2b}=\dfrac{a^2b+ab^2}{?}$

18. $\dfrac{2}{c^2+d^2}=\dfrac{2c^2-2d^2}{?}$.

19. $\dfrac{a+1}{a+2}=\dfrac{a^2+4a+3}{?}$.

20. $\dfrac{a-b-c}{a+c-b}=\dfrac{a^2-(b+c)^2}{?}$

71. **Mixed expressions.** A **mixed expression** is an expression consisting of two parts, one integral, the other fractional.

$2x-\dfrac{3}{ax}$ and $a^2+a-\dfrac{8}{a}$ are mixed expressions.

In arithmetic certain fractions can be reduced to an equivalent integer or a mixed expression by performing the indicated division. Such fractions are called **improper fractions.**

$\dfrac{12}{4}$ and $\dfrac{15}{4}$ are improper fractions. The first equals 3 and the second equals $3\frac{3}{4}$ when the numerator is divided by the denominator.

A **proper fraction** in arithmetic has its numerator less than its denominator.

An **improper algebraic fraction** may be reduced to an integral or a mixed expression by performing the indicated division.

$\frac{a^2+a}{a}$ and $\frac{2a^2+a-1}{a}$ are improper fractions. The first equals $a+1$ and the second equals $2a+1-\frac{1}{a}$ when the indicated divisions are performed.

Exercise 59

Change the following fractions to integral or mixed expressions:

1. $\frac{a^3}{a^2}$.
2. $\frac{7}{2}$
3. $\frac{7a}{2a}$.
4. $\frac{a^3b}{a^2}$.
5. $\frac{3a^3-6a^2}{3a}$.
6. $\frac{4m^2-m}{2m^2}$ Ans. $2-\frac{1}{2m}$.
7. $\frac{4a^3-4a^2+a}{2a^2}$.
8. $\frac{4x^2y-4xy^2+y^3}{2xy}$.
9. $\frac{x^2-y^2-3.}{x-y}$. Ans. $x+y-\frac{3}{x-y}$.
10. $\frac{a^2+3a+4}{a+1}$.
11. $\frac{x^2-2xy+y^2}{x-y}$.
12. $\frac{x^2+y^2}{x+y}$. Ans. $x-y+\frac{2y^2}{x+y}$.
13. $\frac{x^2+y^2}{x-y}$.
14. $\frac{a^2-b^2+4b}{a+b}$.
15. $\frac{9x^2+3xy+3y^2}{3x+y}$.
16. $\frac{a^2-b^2+4ab}{a+b}$.
17. $\frac{a^2-ab+b^2}{a+b}$.

18. $\dfrac{x^3 - 2y^3}{x - y}$. 19. $\dfrac{a^4 + b^4}{a + b}$. 20. $\dfrac{x^3 + y^3}{x + y}$.

21. $\dfrac{x^3 - y^3}{x + y}$. 22. $\dfrac{x^4 + y^4}{x - y}$. 23. $\dfrac{x^4 - y^4}{x^2 + xy + y^2}$.

72. Changing a mixed expression into an equivalent fraction. The rule for changing a mixed expression into an equivalent improper fraction is the same in algebra as in arithmetic.

Rule. *Multiply the integral part by the denominator of the fraction, to this product add or subtract the numerator according to the value of the fraction, and write the result over the denominator.*

Exercise 60

Change the following mixed expressions into equivalent fractions:

1. $1\frac{3}{4}$. Ans. $\dfrac{7}{4}$. 2. $a + \dfrac{b^2}{a}$. Ans. $\dfrac{a^2 + b^2}{a}$.

3. $5\frac{1}{3}$. 4. $x - \dfrac{y^2}{x}$. 5. $x + \dfrac{y^2}{2x}$. 6. $a - \dfrac{b^2}{a}$.

7. $3a + \dfrac{6b}{5}$. 8. $a + 1 - \dfrac{3}{a}$.

9. $4 - \dfrac{x^2}{y^2}$. 10. $x + y + \dfrac{y^2}{x - y}$. Ans. $\dfrac{x^2}{x - y}$.

11. $2x + \dfrac{4xy}{x - 2y}$ 12. $a - b + \dfrac{2b^2}{a + b}$.

13. $m^2 + mn + n^2 + \dfrac{n^3}{m - n}$. Ans. $\dfrac{m^3}{m - n}$.

14. $a^2 - ab + b^2 - \dfrac{b^3}{a + b}$. 15. $x^2 - 2x + 1 + \dfrac{2}{x - 1}$.

73. Addition and subtraction of fractions. If several fractions have a **common denominator,** their algebraic sum may be found by taking the algebraic sum of their numerators and writing it over the common denominator.

For example,

$$\frac{2a}{by}+\frac{3x}{by}-\frac{a}{by}-\frac{x}{by}=\frac{2a+3x-a-x}{by}=\frac{a+2x}{by}.$$

If the fractions to be added do not have a common denominator, it is necessary first to change them to equivalent fractions having the **lowest common denominator** (L. C. D.). The complete operation is as follows:

Rule. *Find the L. C. M. of all the denominators for the L. C. D. Divide this L. C. D. by the denominator of each fraction and multiply both terms of the fraction by the quotient. Divide the algebraic sum of the resulting numerators by the L. C. D.*

Illustrative example:

Find the algebraic sum of $\frac{2x}{3a^2b}+\frac{y}{3ab}-\frac{x}{3b^2}$.

The L. C. D of the denominators is $3a^2b^2$.

$\frac{2x}{3a^2b}=\frac{2bx}{3a^2b^2}$. (Multiplying both terms by what number?)

Similarly, $\frac{y}{3ab}=\frac{aby}{3a^2b^2}$, and $\frac{x}{3b^2}=\frac{a^2x}{3a^2b^2}$.

Therefore, $\frac{2x}{3a^2b}+\frac{y}{3ab}-\frac{x}{3b^2}=\frac{2bx}{3a^2b^2}+\frac{aby}{3a^2b^2}-\frac{a^2x}{3a^2b^2}=$

$\frac{2bx+aby-a^2x}{3a^2b^2}$.

Exercise 61

Add the following:

1. $\frac{1}{4}+\frac{1}{6}$. 2. $\frac{1}{a}+\frac{2}{a}$. 3. $\frac{1}{2a}+\frac{1}{4a}$. Ans. $\frac{3}{4a}$.

4. $\frac{1}{a}+\frac{1}{b}=\frac{b}{ab}+\frac{a}{ab}=\frac{b+a}{ab}$. (Explain.)

5. $\frac{2}{x}+\frac{3}{y}$.

6. $\frac{1}{xy}+\frac{1}{yz}$. Ans. $\frac{z+x}{xyz}$.

7. $\frac{1}{4ab}+\frac{2}{3ab}$.

8. $\frac{1}{a^2b}+\frac{1}{ab^2}$.

9. $\frac{a}{2b^2}+\frac{a^2}{4b}$.

10. $\frac{y}{x}+\frac{x}{y}$. Ans. $\frac{y^2+x^2}{xy}$.

11. $\frac{b}{a^2}+\frac{a}{b^2}$.

12. $\frac{3a}{x^2}+\frac{3b}{y^2}$.

13. $\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2}$.

14. $\frac{1}{x+1}+\frac{1}{x+2}$.

15. $\frac{1}{x-y}+\frac{1}{x+y}$. Ans. $\frac{2x}{x^2-y^2}$.

Subtract in the following:

16. $\frac{1}{x}-\frac{1}{y}$. Ans. $\frac{y-x}{xy}$.

17. $\frac{a}{b}-\frac{b}{a}$. Ans. $\frac{a^2-b^2}{ab}$.

18. $\frac{1}{3}-\frac{2}{9}$.

19. $\frac{a^2}{b}-\frac{b^2}{a}$.

20. $\frac{1}{xy}-\frac{1}{yz}$.

21. $\frac{1}{2a^2}-\frac{1}{3ab}$.

22. $\frac{1}{2a^2b}-\frac{1}{3ab^2}$.

23. $\frac{1}{x-y}-\frac{1}{x+y}$.

Combine the following:

24. $\frac{1}{x}+\frac{1}{y}-\frac{1}{z}$. Ans. $\frac{yz+xz-xy}{xyz}$.

25. $\frac{1}{a}-\frac{1}{b}-\frac{1}{c}$.

26. $\frac{a}{b}+\frac{b}{c}-\frac{c}{a}$.

27. $\frac{3}{4}-\frac{5}{6}+\frac{4}{5}$. Ans. $\frac{43}{60}$.

28. $\dfrac{1}{xy}+\dfrac{1}{yz}-\dfrac{1}{xz}$.

29. $\dfrac{9}{10}-\dfrac{7}{15}+\dfrac{11}{20}$.

30. $1+\dfrac{a}{b}-\dfrac{b}{a}$.

31. $2+\dfrac{x}{y}-\dfrac{y}{x}$.

32. $3+\dfrac{4}{5}-\dfrac{5}{4}$.

33. $\dfrac{x+y}{2a}+\dfrac{x-y}{4a}$.

34. $\dfrac{x+y}{2}-\dfrac{x-y}{4}$. Ans. $\dfrac{x+3y}{4}$.

35. $\dfrac{x+a}{b}-\dfrac{x-b}{a}$. Ans. $\dfrac{ax+a^2-bx+b^2}{ab}$.

36. $\dfrac{m-a}{2b}+\dfrac{m+b}{2a}$.

37. $\dfrac{a+b}{a^2b}-\dfrac{a-b}{b^2a}$.

38. $\dfrac{2a+3b}{2a^2b}-\dfrac{3a-2b}{4ab^2}$.

39. $\dfrac{m-2n}{2am}-\dfrac{n-2m}{2an}$.

40. $\dfrac{a-b}{2a}+\dfrac{b-c}{2b}+\dfrac{c-a}{2c}$.

41. $\dfrac{2y}{x^2-y^2}+\dfrac{1}{x-y}-\dfrac{1}{x+y}=\dfrac{2y}{x^2-y^2}+\dfrac{x+y}{x^2-y^2}-\dfrac{x-y}{x^2-y^2}=\dfrac{4y}{x^2-y^2}$.

42. $\dfrac{y}{4x^2-y^2}-\dfrac{1}{2x-y}$.

43. $\dfrac{2b}{a^2-b^2}-\dfrac{1}{a+b}+\dfrac{1}{a-b}$.

44. $\dfrac{a+2}{a^2-1}+\dfrac{1}{a+1}-\dfrac{1}{1-a}$. Ans. $\dfrac{3a+2}{a^2-1}$.

Note. The L. C. D. of No. 44 is $(a+1)(a-1)$, or a^2-1. This is made possible by changing $1-a$ to $-(a-1)$. Then by multiplying the denominator of the last fraction by -1 and changing the sign of the fraction, we get the following result: $-\dfrac{1}{1-a}=+\dfrac{1}{a-1}$.

The problem now becomes $\dfrac{a+2}{a^2-1}+\dfrac{1}{a+1}+\dfrac{1}{a-1}$.

45. $\frac{b^2}{a^2-b^2}-\frac{a}{b-a}+\frac{b}{a+b}$.

46. $\frac{2}{x+y}-\frac{5}{y-x}+\frac{3}{x^2-y^2}$.

47. $\frac{4y}{x^2-4y^2}-\frac{y}{x+2y}-\frac{y}{2y-x}$.

48. $\frac{a}{a^2-b^2}-\frac{b}{b+a}-\frac{a+b}{a-b}$.

74. Fractional equations. If an equation contains fractions, it may be freed of fractional forms by multiplying both members of the equation by the L. C. M. of the denominators. (This is an application of what axiom?)

The equation $\frac{x}{2}-\frac{x}{3}=7$ becomes $\frac{3x}{6}-\frac{2x}{6}=\frac{42}{6}$, if every term of both members is changed to a fraction whose denominator is the L. C. D. of all the denominators.

$\frac{3x}{6}-\frac{2x}{6}=\frac{42}{6}$ becomes $3x-2x=42$, if both members are multiplied by 6. Therefore $x=42$. Check $\frac{42}{2}-\frac{42}{3}=7$, or $21-14=7$.

Exercise 62

Solve the following and check:

1. $x-\frac{2x}{5}=\frac{6}{5}$. Ans. $x=2$.

2. $\frac{x}{2}-\frac{x}{5}=3$.

3. $\frac{2x}{3}-\frac{x}{4}=5$.

4. $\frac{3x}{4}-\frac{4x}{5}=-1$. Ans. $x=20$.

5. $\frac{x+1}{2}-\frac{3x}{7}=1$.

6. $\frac{x}{2}+\frac{2x}{3}-\frac{18}{6}=4$.

7. $\frac{x+5}{4}-\frac{x+2}{9}=2$.

(**Suggestion.** Multiplying by 36 gives $9x+45-4x-8=72$. Explain the signs $-4x$ and -8).

8. $\dfrac{x-4}{5}+\dfrac{2x-16}{2}=2.$ **9.** $\dfrac{x+4}{3}-\dfrac{7-x}{2}=2.$

10. $\dfrac{2}{x}+\dfrac{3}{x}=5.$

Suggestion. Multiplying by x gives $2+3=5x$. $x=1$.

11. $\dfrac{5}{x}+\dfrac{4}{x}=9.$ **12.** $\dfrac{12}{x}-\dfrac{3}{x}=3.$

13. $\dfrac{9}{a}-\dfrac{5}{a}=2.$ Ans. $a=2.$

14. $\dfrac{3}{5z}+\dfrac{1}{2z}=\dfrac{11}{30}.$ **15.** $\dfrac{3}{4z}-\dfrac{2}{3z}=\dfrac{1}{12}.$

16. $\dfrac{3}{2a}-\dfrac{5}{4a}=\dfrac{1}{2}.$ Ans. $a=\dfrac{1}{2}.$

17. $\dfrac{1}{a}+\dfrac{2}{3a}=\dfrac{5}{12}.$ **18.** $\dfrac{1}{3a}+\dfrac{1}{2a}=\dfrac{5}{6}.$

19. $\dfrac{2}{x+3}=\dfrac{3}{x+6}.$

Hint. $\dfrac{2(x+6)}{(x+3)(x+6)}=\dfrac{3(x+3)}{(x+6)(x+3)}.$

Therefore $2(x+6)=3(x+3)$. Explain.

20. $\dfrac{2}{x-3}=\dfrac{1}{x-2}.$ **21.** $\dfrac{3}{2x+1}=\dfrac{1}{x-1}.$

22. $\dfrac{5}{2x+4}=\dfrac{3}{x+3}.$ **23.** $\dfrac{6}{2x-3}=\dfrac{11}{4x-5}.$

24. $\dfrac{a+2}{a-2}=\dfrac{a+4}{a-1}.$

Hint. $\dfrac{(a+2)(a-1)}{(a-2)(a-1)}=\dfrac{(a+4)(a-2)}{(a-1)(a-2)}.$

Therefore $(a+2)(a-1)=(a+4)(a-2).$

25. $\dfrac{n}{n+2}+\dfrac{3}{n-2}=1.$ Ans. $n=-10.$

26. $\dfrac{4x-3}{2x-1}=\dfrac{4x-7}{2x-5}$. Ans. $x=1$.

27. $\dfrac{6x-2}{3x+4}=\dfrac{2x+1}{x+3}$.

28. $\dfrac{2n+1}{3}+\dfrac{n-4}{2}-\dfrac{n-2}{4}=\dfrac{5n+1}{3}$.

29. $\dfrac{2x-3}{3x-1}=\dfrac{4x-5}{6x}$.

30. $\dfrac{x-1}{6}-\dfrac{x+1}{4}+\dfrac{x-4}{3}=x-16$.

31. $\dfrac{4(2n+7)}{3}-\dfrac{3(2n-1)}{5}=5-n$.

32. $\frac{3}{4}(x+1)-\frac{2}{3}(x+2)+\frac{1}{2}(x+3)=12$.

75. Multiplication of fractions. The product of two or more algebraic fractions is found as in arithmetic.

Rule. I. *Write the indicated product of all the factors of the numerators over the indicated product of all the factors of the denominators.*

II. *Reduce the resulting fraction to its lowest terms by removing all factors common to both terms.*

III. *Take the product of the remaining factors of the numerators for the required numerator, and of the denominators for the required denominator.*

Illustrative example.

Find the product of $\frac{3}{5}$ and $\frac{10}{18}$.

Solution. $\dfrac{3}{5}\times\dfrac{10}{18}=\dfrac{3\cdot 2\cdot 5}{5\cdot 3\cdot 3\cdot 2}=\dfrac{1}{3}$.

Similarly, the product of $\dfrac{a}{b}$, $\dfrac{x}{a^2}$, and $\dfrac{b^3}{x^2}$ is $\dfrac{a\cdot x\cdot b\cdot b\cdot b}{b\cdot a\cdot a\cdot x\cdot x\cdot x}=\dfrac{b^2}{ax^2}$.

Exercise 63

Find the product in each of the following:

1. $\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{5}{6}$.
2. $\frac{a}{b}\cdot\frac{b^2}{c}\cdot\frac{c^2}{d}\cdot\frac{d^2}{a^2}$.
3. $\left(-\frac{3}{4}\right)\cdot\left(-\frac{4}{5}\right)\cdot\left(-\frac{5}{6}\right)$.
4. $\frac{b}{2}\cdot\frac{c}{4}\cdot\frac{12}{ac^2}$.
5. $b\cdot\frac{a}{c}$. Hint. $\frac{b}{1}\cdot\frac{a}{c}$. Ans. $\frac{ab}{c}$.
6. $5\cdot\frac{2}{3}$.
7. $(-10)\cdot\frac{4}{5}$.
8. $\frac{a^2}{b}\cdot\frac{b^2}{a}$.
9. $\frac{a^2}{b}\cdot\frac{b^2}{c^3}\cdot\frac{c^4}{a^3}$.
10. $\frac{a}{b^2}\cdot\frac{b}{c^2}\cdot\frac{c^3}{a^3}$.
11. $\frac{ab}{c}\cdot\frac{cd}{ac}$.
12. $\frac{1}{ab}\cdot\frac{1}{cd}$.
13. $\left(-\frac{a}{bcd}\right)\cdot\left(-\frac{b^2c}{ad}\right)$.
14. $\left(\frac{a^2}{4}\right)\cdot\left(-\frac{9}{a^3}\right)\cdot\left(-\frac{ab}{12}\right)$.
15. $\left(-\frac{x}{5zy}\right)\cdot\left(-\frac{15x^2y^2}{3yz}\right)$.
16. $\left(\frac{x^2}{3}\right)\cdot\left(\frac{3b}{4c}\right)\cdot\left(\frac{4c}{5bx}\right)$.
17. $\left(\frac{x^2-y^2}{12}\right)\cdot\frac{6}{(x-y)}$. Ans. $\frac{x+y}{2}$.

    Hint. $\frac{(x-y)\cdot(x+y)\cdot 2\cdot 3}{2\cdot 3\cdot 2\cdot (x-y)}$.
18. $\frac{a^2-b^2}{c-d}\cdot\frac{c^2-d^2}{a+b}$.
19. $\frac{x^2+2x+1}{4x^2}\cdot\frac{8x}{x^2-1}$.
20. $\frac{2x+1}{a(3-b)}\cdot\frac{9a-ab^2}{4x^2-1}$.
21. $\frac{a^2+4a+3}{a^2+2a+1}\cdot\frac{a^2-1}{a+3}$.
22. $\frac{a^2+4a+4}{a^2+10a+16}\cdot\frac{a^2+16a+64}{a^2-64}$.
23. $\frac{x^2+2xy+y^2}{x^2+xy}\cdot\frac{(x^2-y^2)x}{x^2-2xy+y^2}$.

24. $\dfrac{x^2+ax}{x^2+2ax+a^2} \cdot \dfrac{(x^2-a^2)x}{(x-a)^2ax}$.

25. $\left(\dfrac{a}{b}+1\right) \cdot \left(\dfrac{a-b}{a^2-b^2}\right)$.

Solution. Mixed expressions and integers must be changed to equivalent fractions before the multiplication can be performed.

$\dfrac{a}{b}+1=\dfrac{a+b}{b}$. (Explain).

Therefore,

$$\left(\frac{a}{b}+1\right) \cdot \left(\frac{a-b}{a^2-b^2}\right)=\left(\frac{a+b}{b}\right) \cdot \left(\frac{a-b}{(a-b)\,(a+b)}\right)=\frac{1}{b}.$$

26. $\left(\dfrac{m}{n}-1\right)\left(\dfrac{n}{m^2-n^2}\right)$. Ans. $\dfrac{1}{m+n}$.

27. $\left(\dfrac{2}{3}-x\right)\left(\dfrac{9}{4-9x^2}\right)$.

28. $\left(\dfrac{a}{b}-1\right)\left(\dfrac{ab}{a^2+ab-2b^2}\right)$.

29. $\left(a-\dfrac{b}{a}\right)\left(\dfrac{ab}{b+a}\right)$. **30.** $\left(a-2+\dfrac{1}{a}\right)\left(\dfrac{a}{a^2-1}\right)$.

31. $\left(2-\dfrac{2}{3}\right)\left(2+\dfrac{2}{3}\right)$. **32.** $\left(9-\dfrac{7}{25}\right)\left(\dfrac{25}{2}\right)$.

33. $2\left(1-\dfrac{n}{m}\right)\left(\dfrac{m^2}{m^2-n^2}\right)$.

34. $ab\left(a-\dfrac{b^2}{a}\right)\left(\dfrac{a}{b}-1\right)\left(\dfrac{4}{a^2-2ab+b^2}\right)$.

35. $\left(\dfrac{1}{b^2}-\dfrac{1}{a^2}\right)\left(\dfrac{ab}{a+b}\right)\cdot ab=\left(\dfrac{a^2-b^2}{a^2b^2}\right)\left(\dfrac{ab}{a+b}\right)\dfrac{ab}{1}=$?

36. $\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)\left(\dfrac{abcd}{ac+ab+bc}\right)$.

76. Multiplication of polynomials with fractional coefficients. Such products as $(\frac{1}{2}x - y)\ (\frac{1}{3}x + y)$ may be found by either of the following methods:

I. By direct multiplication.

$$\begin{array}{l} \frac{1}{2}x - y \\ \frac{1}{3}x + y \\ \hline \frac{1}{6}x^2 - \frac{1}{3}xy \\ \qquad + \frac{1}{2}xy - y^2 \\ \hline \frac{1}{6}x^2 + \frac{1}{6}xy - y^2 = \dfrac{x^2 + xy - 6y^2}{6}. \end{array}$$

Explain each partial product. Explain.

II. By considering each factor as a mixed expression.

$$(\tfrac{1}{2}x - y)\ (\tfrac{1}{3}x + y) = \left(\frac{x - 2y}{2}\right)\left(\frac{x + 3y}{3}\right) = \frac{x^2 + xy - 6y^2}{6}.$$

Exercise 64

Find the product in each of the following by both methods:

1. $(\frac{1}{2}a - 3b)\ (3a - \frac{1}{2}b)$.
2. $(\frac{2}{3}a + \frac{1}{6}b)\ (a - \frac{1}{2}b)$.
3. $(a - \frac{3}{2}b)\ (2a + \frac{1}{3}b)$.
4. $\left(\dfrac{x}{2} + 3\right)\left(2x - 16\right)$.
5. $(\frac{1}{2}x + 3)\ (\frac{1}{3}x - 2)$.
6. $(\frac{1}{2}a - \frac{1}{3}b)\ (\frac{1}{2}a + \frac{1}{3}b)$.
7. $(\frac{1}{3}a + \frac{2}{5}b)\ (\frac{1}{3}a - \frac{2}{5}b)$.
8. $(\frac{1}{2}a - \frac{2}{3}b)\ (a + \frac{1}{2}b)$.
9. $\left(a - \frac{1}{2}b\right)\left(\dfrac{a}{3} + \dfrac{b}{2}\right)$.
10. $\left(x - \frac{1}{2}y\right)\left(\dfrac{x}{2} - \dfrac{y}{4}\right)\left(\dfrac{2x + 2y}{3}\right)$.
11. $\left(\dfrac{a}{2} - 1\right)\left(\frac{1}{2}a + 1\right)\left(\dfrac{4}{a^2 - 4}\right)$.
12. $\left(\dfrac{a}{2} - 1\right)\left(\dfrac{a^2}{4} + \dfrac{a}{2} + 1\right)$.
13. $\left(\dfrac{a}{3} - \dfrac{b}{2}\right)\left(\frac{1}{9}a^2 + \frac{1}{6}ab + \frac{1}{4}b^2\right)$.

77. Division by a fraction. In division by an algebraic fraction the process is the same as in arithmetic.

Rule. *Invert the divisor and proceed as in multiplication.*

Illustrative examples.

1. Divide $\frac{2}{3}$ by $\frac{5}{6}$. $\frac{2}{3} \div \frac{5}{6} = \frac{2}{3} \cdot \frac{6}{5} = \frac{4}{5}$.

2. Divide $\frac{a}{b^2}$ by $\frac{c}{b}$. $\frac{a}{b^2} \div \frac{c}{b} = \frac{a}{b^2} \cdot \frac{b}{c} = \frac{a}{bc}$.

3. Divide 2 by $\frac{2}{3}$. $\frac{2}{1} \cdot \frac{3}{2} = 3$.

Observations. If the divisor is an integer it should be put in fractional form. As, for example, to divide $\frac{2}{3}$ by 2, we multiply $\frac{2}{3}$ by $\frac{1}{2}$, etc.

If the divisor is a mixed expression, it should be changed to an equivalent fraction.

Exercise 65.

Perform the following divisions:

1. $4 \div \frac{4}{5}$.
2. $8 \div \frac{2}{15}$.
3. $9 \div \frac{3}{11}$.
4. $\frac{9}{10} \div 3$.
5. $\frac{8}{13} \div 4$.
6. $\frac{a^2}{b^2} \div \frac{a}{b}$.
7. $\frac{6}{11} \div \frac{2}{3}$.
8. $\frac{6}{11} \div \frac{3}{2}$.
9. $\frac{x^2}{y} \div x$.
10. $\frac{x^2}{y^2} \div \frac{x}{y}$.
11. $\frac{x^2}{y^2} \div \frac{y}{x}$.
12. $a \div \left(-\frac{b}{a}\right)$.
13. $\frac{x^2 - y^2}{xy} \div \frac{x + y}{y}$.
14. $\left(\frac{a^2}{b^2} - 1\right) \div \left(\frac{a}{b} - 1\right)$.

Hint. $\frac{a^2}{b^2} - 1 = \frac{a^2 - b^2}{b^2}$, and $\frac{a}{b} - 1 = \frac{a - b}{b}$.

15. $\frac{a + 1}{2} \div \frac{3}{a + 1}$.
16. $\frac{a + 1}{2} \div \frac{a + 1}{4}$.

17. $\left(\frac{x}{a}+\frac{y}{b}\right)\div\left(\frac{bx+ay}{a}\right)$. **Hint.** $\frac{x}{a}+\frac{y}{b}=\frac{bx+ay}{ab}$.

18. $\left(\frac{1}{a}-\frac{1}{b}\right)\div\left(\frac{1}{a}+\frac{1}{b}\right)$.

19. $\left(\frac{x}{y}-\frac{y}{x}\right)\div\left(\frac{x-y}{xy}\right)$.

20. $\frac{a}{b}\div\frac{b}{c}\div\frac{c}{d}$.

Hint. Inverting each divisor gives $\frac{a}{b}\cdot\frac{c}{b}\cdot\frac{d}{c}$.

21. $\frac{2}{3}\div\frac{3}{4}\div\frac{4}{5}\div\frac{5}{6}$. Ans. $\frac{4}{3}$.

22. $\frac{2a}{3b}\div\frac{4a^2}{9b^2}\div\frac{3b}{8a}$.

23. $\frac{2m}{9n}\div 4m\div\frac{5}{3n}$.

24. $\frac{12xy^2}{13ab}\div\frac{15x^2y}{26a^2b}\div 4b$.

25. $\frac{10n}{7m}\div\frac{m+n}{5n}\div 21m$.

26. $\frac{x^2-y^2}{x^2}\div\frac{x^2+2xy+y^2}{xy}\div\frac{x^3-x^2y}{xy+y^2}$.

27. $\frac{a^2-a-2}{a^2-3a-10}\div\frac{a^2-1}{a^2-10a+25}\div\frac{a^2-7a+10}{a^2+3a+2}$.

28. $\frac{1}{(x-3)\,(x-4)}\div\frac{2}{x^2-x-12}\div\frac{4}{x^2-9}$.

29. $\left(\frac{1}{1-a}+\frac{1}{1+a}\right)\div\left(\frac{1}{1-a}-\frac{1}{1+a}\right)$.

Hint. $\frac{1}{1-a}+\frac{1}{1+a}=\frac{2}{1-a^2}$.

30. $\left(x+3+\frac{5}{x-3}\right)\div\left(x-3+\frac{5}{x+3}\right)$.

31. $\left(\frac{x+y}{y}+\frac{y}{x+y}\right)\div\left(\frac{1}{x}+\frac{1}{y}\right)$.

Exercise 66. Review

1. If $a=2$ and $b=-5$, what is the value of $-3a^2b^3$?
2. Divide $6x^2+x-7$ by $x-1$.
3. Divide $n^4-5n^3+11n^2-12n+6$ by n^2-3n+3.
4. Divide $a^3+b^3+3ab^2+3a^2b$ by $a^2+2ab+b^2$.
5. If $x-2$ is one factor of x^3-8, what is the other?
6. If x^2-xy+y^2 is one factor of $x^4+x^2y^2+y^4$, what is the other?
7. Divide $64a^3-27b^3$ by $4a-3b$.
8. Multiply a^2-ab+b^2 by $a^3-3a^2b+3ab^2-b^3$.
9. Simplify $13x-2(-x+3)+3(3x-5)-8$.

Find the value of the literal number in each of the following equations. If the equation reduces to the quadratic form it must be solved by the factoring process.

10. $(2x+1)^2=4(x-2)(x+7)+25$.
11. $(x+3)^2-(x+2)(x+5)=x+1$.
12. $x^2+(x+1)^2+(x+2)^2=29$.
13. $3x-2(2x-5)-3(7-x)=-15$.
14. $(x+3)^2-(x-2)(x-5)=12x+4$.
15. $(x-2)^2+(x-4)^2=100$.
16. $(3x-5)^2-3(x+3)(3x+1)=196$.
17. $(x-1)^2+(x-2)^2+(x-3)^2=50$.
18. $\dfrac{n+1}{5}+\dfrac{n-1}{3}-\dfrac{n-5}{2}=n-16$.
19. $\dfrac{5}{x-2}-\dfrac{3}{x+2}=\dfrac{x+13}{(x-2)(x+2)}$.

Simplify the following:

20. $\dfrac{x+1}{x^2-5x+6}+\dfrac{x-1}{x^2-9}-\dfrac{x+2}{x^2+x-6}$.

21. $\dfrac{3n^2}{n^2-9n}\cdot\dfrac{3n^2+9n}{n^2-7n+12}\div\dfrac{9n^3}{n^2-6n+9}$.

22. State the method for finding the H.C.F. of two or more algebraic expressions, and find it for $6a^2b^2$, $12a^3b^2$, and $16a^3bc^2$.

23. Find the H.C.F. of $3a^3+7a^2b-20ab^2$, $a^4-16a^2b^2$, and a^4+4a^3b.

24. Explain how the L.C.M. of two or more algebraic expressions is obtained.

In what operations have you found it necessary to use the L.C.M.? Write one exercise illustrating each case.

25. Separate 14 into two parts such that their quotient is 6.

Suggestion. Let x represent one part and $14-x$ the other.

26. Separate 48 into two parts such that two-thirds of the greater shall equal twice the less.

27. What number must be subtracted from both terms of the fraction $\frac{11}{14}$ so that the resulting fraction shall equal $\frac{1}{2}$?

28. Find three consecutive numbers such that one-half of the first plus one-third of the second minus one-fourth of the third equals 8.

29. Two numbers differ by 6 and one-half of the larger exceeds one-third of the smaller by 10. Find the numbers.

30. A merchant starting in business spends one-sixth of his capital for furniture and equipment, one-half of it for stock, and one-twelfth for rent, insurance, and advertising. If he has $3,000.00 left with which to operate his business, find his original capital.

31. The same number is added to both the numerator and the denominator of the fraction $\frac{5}{9}$. What is the number if the resulting fraction is equal to $\frac{2}{3}$?

32. Find three numbers such that the second is 2 more than 3 times the first and the third is 2 less than 5 times the first. If one-sixth of the third number be subtracted from one-half of the second, the remainder is 4.

33. Eight years ago A was one-fourth as old as B, but in 12 years from now he will be one-half as old as B. Find their ages now.

34. The sum of two numbers is 40. Two-thirds of the larger exceeds one-half of the smaller by 8. Find the numbers.

35. The sum of two numbers is 15 and their product is 50. Find the numbers.

36. Find two consecutive integers whose product is 42.

37. The sum of a number and its reciprocal is $\frac{10}{3}$. What is the number?

Note. The reciprocal of a number is the quotient of 1 divided by the number.

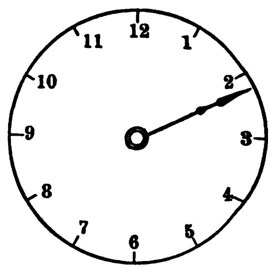
38. The sum of the reciprocals of two consecutive even numbers is $\frac{5}{12}$. What are they?

39. James had one-half as many marbles as John; but after John had given him 2 marbles, he had three-fourths as many as John. How many had each at first?

40. Divide 36 into two parts such that the larger divided by the smaller will equal the smaller divided by 3.

41. The numerator of a certain fraction is x and the denominator is 2 more than the numerator. The value of the fraction is the same as the result obtained by subtracting x from both terms of $\frac{15}{23}$. Find x and the first fraction.

42. At what time between 2 and 3 o'clock are the hands of the clock together?



Explanation. At 2 o'clock the hour hand is at 2 and the minute hand is at 12. At some time between 10 minutes past 2 and 15 minutes past 2 the minute hand will overtake and pass the hour hand. The unit of distance in this and all similar problems is the minute-space, which is the distance passed over by the minute hand in one minute. It is apparent that, starting at 2 o'clock, the minute hand must travel 10 minute-spaces more than the hour hand in order to overtake the hour hand. It is also apparent that the minute hand travels 12 times as fast as the hour hand.

Solution. Let x represent the number of minute-spaces the minute hand moves before they are together.

And $\frac{x}{12}$ = the number of minute-spaces the hour hand travels.

$x - \frac{x}{12} = 10.$ Whence $x = 10\frac{10}{11}$. Ans. $10\frac{10}{11}$ minutes past 2.

43. At what time between 3 and 4 o'clock are the hands of a clock together?

44. At what time between 2 and 3 o'clock are the hands of a clock exactly opposite each other?

Suggestion. Starting at 2 o'clock, the minute hand must gain in all 40 minute-spaces. Ans. $43\frac{7}{11}$ minutes past 2.

45. At what time between 3 and 4 o'clock are the hands of a watch opposite each other?

46. At what time between 12 and 1 o'clock are the hands of a clock opposite each other?

47. At what time between 8 and 9 o'clock are the hands of a clock opposite each other?

48. At what time between 10 and 11 o'clock are the hands of a clock together?

RATIO AND PROPORTION

78. Ratio. In arithmetic the **ratio of two numbers** gives in simplest form the relation between them when they are compared as to size, or magnitude. It is that **multiple** of the second number that gives the first as a product and is found by dividing the first by the second.

The first number of a ratio is known as the **antecedent** and the second number as the **consequent.**

The **symbol** of ratio is the **colon** (:) or the **bar** of the fraction.

The expression 6 in. : 24 in. is to be read "the ratio of 6 in. to 24 in." and its value is that of the fraction $\frac{6}{24}$ or $\frac{1}{4}$. In other words, the ratio of 6 inches to 24 inches is equal to the ratio of 1 to 4, or 6 in. $= \frac{1}{4} \cdot 24$ in.

A ratio is treated as a fraction and is simplified by reducing it to its lowest terms.

If two **denominate** numbers are to be compared in a ratio they must be in units of the same kind, or reducible to units of the same kind.

For instance, the ratio of 8 inches to 1 yard can be found only when 1 yard is reduced to inches. Then, 8 in. : 1 yd. is the same as 8 in. : 36 in., 2 : 9.

The ratio of two **abstract** numbers is found by dividing the first by the second.

Exercise 67

1. Find the ratio of 3 bushels to 6 bushels.

Solution. 3 bu. : 6 bu. is the same as 3 : 6, or 1 : 2. (May also be written $\frac{1}{2}$.)

2. Find the ratio of 4 qt. to 2 bu.

Solution. Since 2 bu. = 64 qt., then 4 qt. : 2 bu. is the same as 4 qt. : 64 qt., or 1 : 16.

3. Find the ratio of 9 to 24, of 10 to 35, of 15 to 36.

4. What is the ratio of 2 in. to 1 ft.?

5. What is the ratio of 4 yd. to 7 ft.?

6. What is the ratio of 100 yd. to 1 mile?

7. What is the ratio of 75 lb. to 1 ton? of 4 tons to 250 lb.?

8. What is the ratio of 10 qt. to 4 gal.? of 25 pt. to 8 qt.? of 10 pt. to 10 gal.?

9. What is the ratio of 1 foot to 1 rod? of 1 foot to 1 inch?

10. What is the ratio of 1 foot to 1 yard? to 1 mile?

11. What is the ratio of 1 inch to 1 foot? 1 yard? 1 rod? 1 mile?

12. If a foot rule is cut into two parts on the line marked 8 inches, what is the ratio of the two parts? What is the ratio if it is cut on the 7 inch line?

13. If a yard stick is cut into two parts on the line marked 15 inches, what is the ratio of the two parts?

14. State the antecedent and the consequent of each of the ratios of the preceding exercises.

Note. The ratio of two algebraic expressions is found by dividing the first by the second and then reducing the result, if a fraction, to its lowest terms.

15. What is the ratio of $4abc$ to $10abc$? of $4abc$ to $10bcd$?

16. What is the ratio of $9a^2b^3c$ to $21b^3c^2d$?

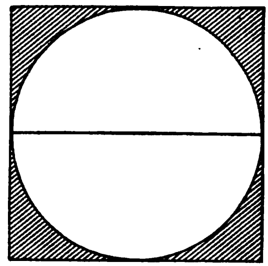
17. What is the ratio of $18x^2y^2z$ to $24x^3y^3z^2$?

18. What is the ratio of $x^2 - y^2$ to $x^2 - xy$?

19. What is the ratio of the circumference of a circle to its radius?

Suggestion. Since $c = 2\pi r$, then $c : r$ equals $2\pi r : r$. Reduce the resulting ratio to lowest terms.

20. What is the ratio of the side of a square to its perimeter?



21. What is the ratio of the diameter of a circle to its circumference?

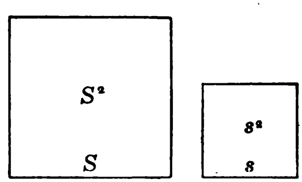
A circle is said to be **inscribed** in a square when it touches each side of the square at its middle point. (See accompanying figure.) The square is said to be **circumscribed** about the circle.

22. Find the ratio of the perimeter of a square to the circumference of its inscribed circle.

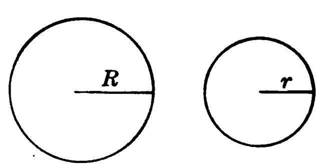
Suggestion. Notice that the diameter of the circle equals one side of the square and that the radius is one-half of a side.

23. What is the ratio of the area of a square to the area of its inscribed circle?

24. Notice that the circle divides the area of the square into five parts. What is the ratio of the area of one of the small parts to the area of the circle?



Suggestion. The area of one of the small parts can be found by subtracting the area of the circle from the area of the square and dividing the remainder by 4. The ratio of the area of one of these parts to that of the circle is $\frac{4-\pi}{4\pi}$.



25. The sides of two squares are S and s. Find the ratio of their perimeters; of their areas.

26. The radii of two circles are R and r. Find the ratio of their circumferences; of their areas.

27. Find the ratio of the circumferences of two circles, if the radius of the second is twice that of the first. Also find the ratio of their areas.

28. Find the ratio of the circumferences of two circles, if the radius of the second is n times that of the first. Find the ratio of their areas.

29. Find the ratio of the areas of two squares, if the side of the second is twice that of the first; if it is 3 times that of the first; if it is n times that of the first.

79. Proportions. The expression of **equality between two ratios** that are known to be equal by the sign = is called **a proportion.**

If $a : b$ is known to equal $c : d$, the relation may be written $a : b = c : d$, or better, $\frac{a}{b} = \frac{c}{d}$.

In either case the expression is to be read, "the ratio of a to b equals the ratio of c to d," or in shorter form, "a is to b as c is to d."

The four numbers, a, b, c, and d, are called the **terms of the proportion.** The first and last terms, a and d, are called the **extremes,** and the second and third terms, b and c, are called the **means** of the proportion.

A **proportion** is an **equation,** and therefore subject to all the axioms of the equation.

From the proportion $\frac{a}{b} = \frac{c}{d}$, we derive the equality $ad = bc$.

(Notice that we cleared the equation of fractions by multiplying both members by bd.)

Since a, b, c, and d are any four numbers in proportion, we have the following:

Rule. *In any proportion the product of the extremes is equal to the product of the means.*

Exercise 68

Test each of the following to determine whether it is a correct proportion:

1. $3:4=9:12$.
Test. Does $3\cdot12=9\cdot4$?

2. $4:6=8:12$.

3. $5:12=25:60$.

4. $3:8=12:32$.

5. $3:8=18:42$.

6. $6:9=12:18$.

7. $\frac{12}{21}=\frac{21}{35}$.

8. $\frac{24}{36}=\frac{35}{54}$.

9. $\frac{10}{45}=\frac{25}{115}$

10. $\frac{20}{35}=\frac{30}{48}$.

11. $\frac{3}{12}=\frac{25}{10}$.

12. $\frac{\frac{1}{3}}{\frac{1}{4}}=\frac{12}{9}$.

Find the value of the literal number in each of the following:

13. $x:5=4:10$.
Ans. $x=2$.

14. $a:6=8:3$.

15. $5:n=12:24$.

16. $6:12=a:18$.

17. $7:21=b:3$.

18. $8:24=5:y$.

19. $\frac{12}{18}=\frac{9}{x}$.

20. $\frac{15}{21}=\frac{b}{42}$.

21. $\frac{25}{35}=\frac{6}{n}$.

22. $\frac{25}{40}=\frac{12}{m}$.

23. $\frac{18}{25}=\frac{x}{30}$.

24. $\frac{x}{x-3}=\frac{8}{5}$.

25. $\frac{x-5}{2x-1}=\frac{2}{5}$.

26. $\frac{x-3}{x-4}=\frac{x+5}{x+7}$.

27. What number is in the same ratio to 6 that 5 is to 3?
Hint. $\frac{x}{6}=\frac{5}{3}$.

28. What number is in the same ratio to 10 that 12 is to 5?

29. How may a foot rule be cut into two parts so that the two parts shall be in the ratio 1 : 2?

Solution. Let $x =$ the number of inches in one part,
and $12 - x =$ the number of inches in the other part.

Then $\frac{x}{12 - x} = \frac{1}{2}$.

It will be noticed that the rule may be cut on the cross line marked 4 inches or on the line marked 8 inches.

30. How may a foot rule be cut into two parts so that the parts shall have the ratio 1 : 3? 3 : 5?

31. How may a yard stick be cut so that the two parts shall have the ratio 4 : 5? 5 : 7? 7 : 11? 3 : 5?

32. The concrete in a wall is a mixture of one part cement and five parts sand. How much is there of each in a wall that measures 2,400 cu. ft.?

33. Two partners divide $4,000 between them so that the shares are in the ratio 3 : 5. What does each receive? If the ratio were 3 : 7, what would each receive?

34. The numbers 10, 23, 50, and x are in proportion in the order named. Find x.

35. Divide a line 20 inches long into two parts that are to each other as 3 : 7.

36. Divide the number 50 into two parts that have the same ratio as 9 : 16.

37. How many true proportions can you write using the numbers, 2, 4, 6, and 12?

38. What number must be subtracted from each of the numbers, 8, 9, 11, and 13 so that the remainders shall be in proportion?

39. The ratio of the weights of two boys is $\frac{7}{8}$. If the larger boy weighs 96 pounds, what does the smaller boy weigh?

CHAPTER VI

SIMULTANEOUS EQUATIONS AND GRAPHING.

80. Equations with two or more unknowns. In the preceding chapters the student has learned how to solve formal equations with no more than one unknown number, and all verbal problems have been such as could be translated into equations with a single unknown.

The solution of a problem often requires the use of several unknowns which are usually dependent one on another.

Study carefully the following types:

I. Problems dealing with the cost of an article.

For example, the cost of a railroad ticket requires a knowledge of the rate in cents per mile and the number of miles to be traveled. In the form of an equation we write,

$c = rn$, where c is the total cost of the ticket in cents, r is the rate in cents per mile, and n is the number of miles to be traveled.

II. Problems involving time, rate, and distance.

The distance that an automobile may travel depends on what its rate is in miles per hour and the number of hours that it runs.

In equation form we write, $d = rh$.

What are d, r, and h in this equation?

III. Problems arising from finding interest.

The interest due on a certain note depends upon the number of dollars for which the note was drawn, the interest rate per cent, and the time since the note was made.

In equation form we write, $i = prt$.

What are i, p, r, and t in this equation?

IV. Problems concerning work done.

The number of bricks that a brick layer may place in a wall will depend upon how many he may be able to lay each hour and the number of hours that he works.

State this in equation form telling what each literal number represents.

81. Definitions. In I, if r is 3, the equation becomes $c=3n$, where the value of c is an exact multiple, 3, of the value of n.

In II, if r is 20, the equation becomes $d=20h$.

Since h, and therefore d, are constantly changing while the car is in motion, they are called **variables.** The rate, 20 miles per hour, is fixed for this particular problem and is called a **constant.**

A **variable** is an algebraic number that is continually changing its value throughout a given problem.

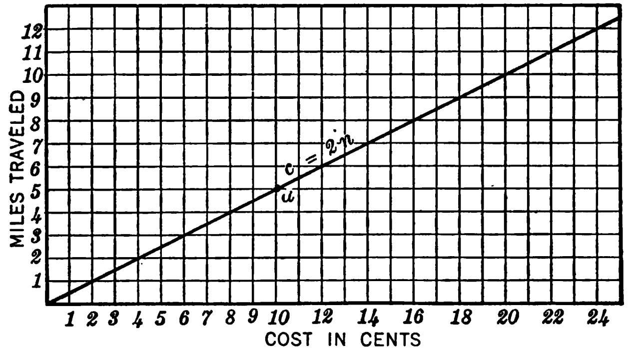
A **constant** is a number that keeps a particular value throughout a given problem.

When one **variable depends** upon another for its value, it is said to be a **function** of the other.

In the equation $c=3n$, the variable c is a function of, or depends upon, the variable n. Similarly in the equation $d=20h$, d is a function of h. In the equation $i=prt$, if p is given the definite value of 400 dollars, and r the definite rate of 6%, then i is a function of t, the time.

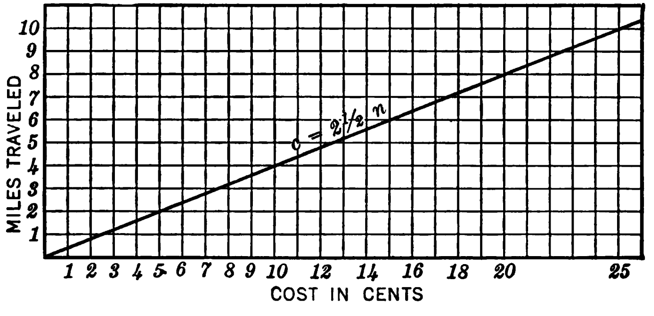
Evidently the word "function" is used to name the idea that one unknown has a definite value for each value given some other unknown number in the equation in which the two are used.

82. The graph of an equation with two variables. When one variable is a function of another it is possible by using cross section paper to picture, or graph, the equation which expresses the relationship between the two.



A study of the above graph of $c = 2n$ will serve to show how a ticket seller might determine the cost of a railroad ticket at 2 cents per mile. The cost of a ticket for five miles is found directly below the point (a) where the line marked 5 miles crosses the graph line, $c = 2n$, and is 10 cents. Find the cost of an 8 mile ticket; of a 10 mile ticket.

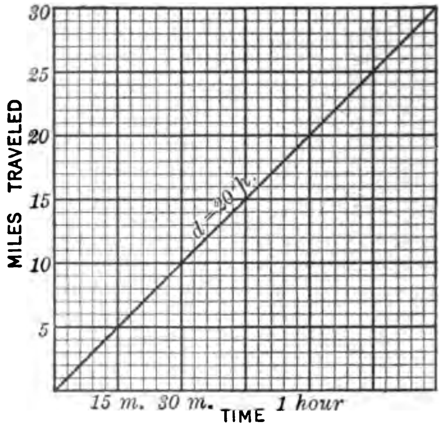
Such a graph, if constructed on a large scale, would enable him to determine quickly the cost of a ticket for any number of miles and, if the rate were $2\frac{1}{2}$ cents, would save considerable computation.



On the graph of $c = 2\frac{1}{2}n$, note that the cost of a 5 mile ticket is $12\frac{1}{2}$ cents. Read the cost of a 10 mile ticket; of an 8 mile ticket; of a 7 mile ticket.

On a large sheet of cross section paper similar to that used for the graph of $c = 2\frac{1}{2}n$, construct the graph when the rate is $3\frac{1}{2}$ cents per mile, or $c = 3\frac{1}{2}n$, and read the cost of a ticket for 5 miles; for 12 miles; for 18 miles.

Study carefully the graph of the equation $d = 20h$. (See II, §80.) The divisions along the horizontal line are named in half hours and hours and therefore each fine division represents 3 minutes. The divisions along the vertical line at the left represent 5 mile spaces and each fine division represents 1 mile. To find the number of miles traveled in 15 min., move the pencil upward on the vertical line representing 15 min., until the graph is reached and then read the number of miles at the left, or 5.



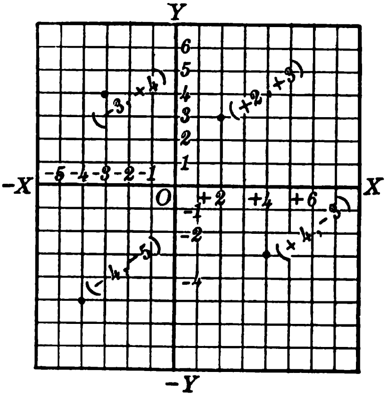
Exercise 69

1. On the graph of $d = 20h$, read the distance traveled in $1\frac{1}{2}$ hours; in 45 min.; in 75 min.; in 54 min.

2. Construct the graph of $d = 15h$ and read the distance traveled in 2 hours; in 12 min.; in 20 min.

3. Find from the graph in No. 2 how many minutes will be required to go 8 miles; 20 miles; 25 miles.

83. Location of a point on cross section paper. On a sheet of cross section paper, with ruler and pencil reinforce one convenient horizontal line, from $-X$ to X, naming it the X axis, and one vertical line, from $-Y$ to Y, naming it the Y axis, as in the accompanying figure.



These lines are called the **axes of co-ordinates.**

Name the point of their intersection 0 (Zero).

Consider the uniform space between consecutive parallel lines as the unit of distance and number the successive intersections along the X axis to the right from 0 as $+1$, $+2$, $+3$, etc., and to the left as -1, -2, -3, etc. Notice that this notation is in agreement with the figure used in the discussion of positive and negative numbers.

Similarly, number the intersections on the Y axis above 0 as $+1$, $+2$, $+3$, etc., and below 0 as -1, -2, etc.

To locate a certain point it is agreed that two values be given, the first to tell the number of spaces which must be counted to the right or left of 0, and the second to tell the number to be counted up or down. To locate the point $(+2, +3)$ count 2 spaces to the right and up on the vertical line at that point 3 spaces. (Note the point in the figure.) To locate the point $(-3, +4)$, count 3 spaces to the left and then up 4 spaces.

The number of spaces counted to the right or left from 0 is known as the **abscissa** of the required point, and the number of spaces up or down, as the **ordinate.**

Exercise 70

Locate the following points:

1. $(-1, +12)$.	**5.** $(-9, +7)$.	**9.** $(0, +4)$.
2. $(+5, -8)$.	**6.** $(+8, +8)$.	**10.** $(-3, 0)$.
3. $(-8, +11)$.	**7.** $(-6, -6)$.	**11.** $(0, 0)$.
4. $(-6, -7)$.	**8.** $(+4, -8)$.	**12.** $(-8, -1)$.

13. Locate the points $(+4, -3)$ and $(+4, +3)$ and tell how far the first is from the second.

14. How far is the point $(+4, +4)$ from the point $(-4, +4)$?

15. Connect the point $(+4, +4)$ with the point $(-4, -4)$ by a straight line and notice that this line passes through the point $(+3, +3)$.

Name four other points on this line.

16. Connect by a straight line the points $(0, +6)$ and $(+6, 0)$ and name five other points on this line.

84. The graph of an equation of the first degree in two unknowns.

The equation $x + y = 5$, when placed in the form $x = 5 - y$ (how may this be done?) shows that x is a function of y, for the value of x will depend upon what value is given y.

If put in the form $y = 5 - x$, y is a function of x.

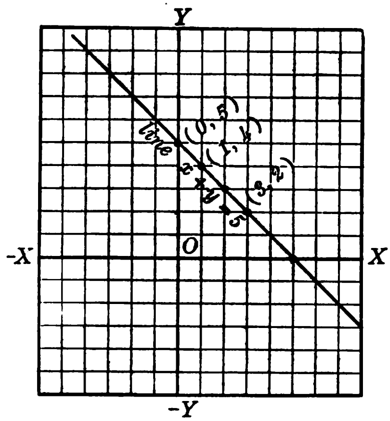
Taking the form $x = 5 - y$, $x = 3$ when $y = 2$, $x = 1$ when $y = 4$, $x = 0$ when $y = 5$. Clearly there is a definite value of x for each value given to y.

Similarly, taking the form $y = 5 - x$, $y = 3$ when $x = 2$, $y = 0$ when $x = 5$, and there is a definite value of y for each value given x.

These pairs or sets of values may be tabulated as follows:

	$x=3$	2	1	0	-1	-2	-3	-4	-5	-6
if	$y=2$	3	4	5	6	7	8	9	10	11

Using the value of x as the abscissa and the corresponding value of y as the ordinate, locate the successive points $(+3, +2)$, $(+2, +3)$, $(+1, +4)$, etc., as in the accompanying figure and notice that these points all lie along a straight line. Draw this line. Every possible pair of values, integral or fractional, that satisfies the equation $x + y = 5$ will be found to lie on this line. Try several fractional values.



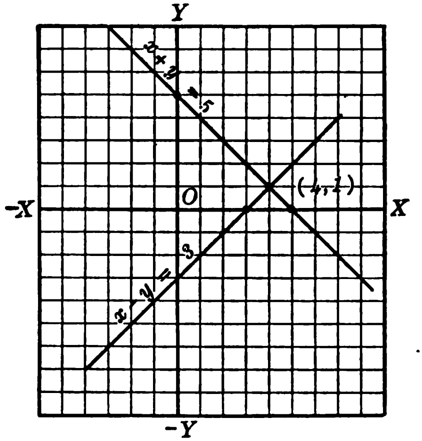
The graph of each and every equation of the first degree in two unknowns is a straight line, and this fact gives rise to the name, **linear equation,** or the **equation of a straight line.**

Exercise 71

On large sheets of cross section paper graph each of the following, finding at least four pairs of values:

1. $2x + y = 4$.
2. $2x + 3y = 6$.
3. $x - y = 3$.
4. $3x - 2y = 6$.
5. $3x - 2y = 12$.
6. $4x + 3y = 12$.
7. $2x - 5y = 10$.
8. $x + 7y = 14$.
9. $2x - 3y = 6$.
10. $4x - 6y = 12$.

11. Place the graphs of $x + y = 5$ and $x - y = 3$ on the same axes and notice their point of intersection.



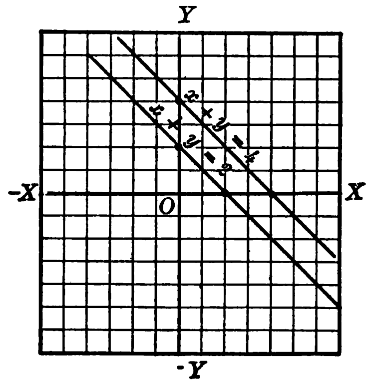
12. Where does the line $x + y = 4$ intersect the line $x - y = 2$ if both are placed on the same axes?

85. Definitions. Since each linear equation in two unknowns has a limitless number of sets of values that satisfy it, evidently a single equation taken alone will be of little service in solving a problem. For that reason a single linear equation is said to be **indeterminate.**

In the figure for No. 11 of Exercise 71, the two lines $x + y = 5$ and $x - y = 3$, intersect at the point (4, 1) and this particular set of values **satisfies both equations.**

If two linear equations involve the same two unknowns, they are said to form a **system of equations**, and, if both equations are satisfied by the same values of the unknowns, the system is called a **simultaneous system.** The system, $x + y = 5$ and $x - y = 3$, is a simultaneous system with a single set of values, $x = 4$ and $y = 1$, satisfying both equations.

Not all linear systems in the same two unknowns are simultaneous, $x+y=2$ and $x+y=4$ have no common pair of values, and if graphed on the same sheet, their lines will be found to be parallel. Such systems are called **contradictory,** or **incompatible.**



Some systems have many pairs of values common to both equations. $x-2y=5$ and $3x-6y=15$ have a limitless number of pairs of values in common and, if graphed, the lines will be found to coincide. The second equation can be obtained from the first by multiplying both members by 3, and is called a **dependent equation.**

Two equations are **independent** when one cannot be obtained from the other by any process that does not destroy the values of the unknowns.

Exercise 72

Graph each of the following systems and determine which are contradictory, independent, and simultaneous.

1. $2x-y=8$
   $x+y=7.$

2. $2x+\ y=4$
   $4x+2y=14.$

3. $2x-3y=18$
   $4x-6y=36.$

4. $3x-2y=7$
   $x+3y=17.$

5. $x-4y=12$
   $2x+2y=24.$

6. $x+3y=10$
   $3x+9y=30.$

86. Formal solution of a simultaneous linear system. Since a system of independent linear equations in two unknowns has one and only one pair of values that satisfies both equations, evidently the solution of such a system consists in finding this common pair of values.

While the solution by means of graphs gives a clear explanation of the relation between the equations, it does not give the exact results if the values are fractional, and the process requires careful and tedious work to get satisfactory results.

Formal algebra gives several methods as follows:

87. Method I. Elimination of one of the unknowns by addition or subtraction.

Example 1. $x - y = 1$ (1) (For convenience we will name
$x + y = 5$ (2) the equations (1) and (2).)

If we add the right members of (1) and (2) together and the left members together, the two sums will be equal under what axiom? This addition of (1) and (2) gives the new equation $2x = 6$. Therefore $x = 3$. Substituting 3 for x in (1) gives the equation $3 - y = 1$. Solving this equation, $y = 2$.

Substituting those values of x and y in both equations, gives $3 - 2 = 1$ and $3 + 2 = 5$. Therefore $x = 3$ and $y = 2$ is the pair of values required.

Example 2. Given $4x - 3y = 11$ (1)
$3x + 2y = 21$ (2)

Multiplying both members of (1) by 2, $8x - 6y = 22$ (3)

Multiplying both members of (2) by 3, $9x + 6y = 63$ (4)

Adding (3) and (4), $17x = 85$. Therefore $x = 5$.

Substituting 5 for x in (2), $y = 3$. Checking in both (1) and (2), $x = 5$ and $y = 3$ satisfies both equations.

Rule. *To solve a set of simultaneous equations of two unknowns, multiply the members of both equations by such numbers as will give equal absolute values for the coefficients of one of the unknowns.*

Add or subtract the resulting equations to eliminate one of the unknowns.

Exercise 73

Solve the following systems by addition or subtraction and check:

1. $2x - y = 9$ $2x + y = 11.$	**2.** $3x - 2y = 1$ $2x + 3y = 18.$	**3.** $4x + 3y = 2$ $5x - 2y = 14.$
4. $x - 3y = 7$ $x - 5y = 7.$	**5.** $2x + y = 5$ $3x + 3y = 3.$	**6.** $4x + 3y = 1$ $2x + 4y = 8.$
7. $2x + 3y = 7$ $6x - y = 1.$	**8.** $3x - 6y = 6$ $4x - 10y = 7.$	**9.** $2x + 8y = 3$ $4x + 4y = 2.$
10. $x + y = 0$ $3x - 2y = 10.$	**11.** $6x + 4y = 4$ $9x - 2y = 2.$	**12.** $3x + 2y = 5$ $9x - 4y = 0.$
13. $2a + 3b = 2$ $6a - 9b = 0.$	**14.** $6c + 2d = 4$ $d - 9c = 4.$	**15.** $2b + c = 2$ $4b - 6c = 8.$

88. Method II. Elimination of one of the unknowns by substitution.

Example 1. Given

$$\begin{aligned} 2x + 4y &= 12 \quad (1) \\ 3x - 2y &= 10 \quad (2) \end{aligned}$$

From (1), $2x = 12 - 4y$ and $x = 6 - 2y$.

Substituting this value for x in (2), $3(6 - 2y) - 2y = 10$.

Solving, $y = 1$. Substituting this value of y in (1), $x = 4$.

Check these values in both equations.

Example 2. Given

$$\begin{aligned} 3x + 2y &= 7 \quad (1) \\ 5x - 3y &= -1. \quad (2) \end{aligned}$$

From (1) $3x = 7 - 2y$ and $x = \dfrac{7 - 2y}{3}$.

Substituting this value of x in (2), $\dfrac{5(7 - 2y)}{3} - 3y = -1$.

Solving, $y = 2$. Substituting this value of y in (1), $x = 1$.

Check these values in both equations.

Rule. *Solve one of the equations for one unknown in terms of the other. Substitute the result for that unknown in the other equation.*

Exercise 74

Solve the following systems by substitution and check:

1. $n + 2y = 8$, $3n - y = 3$.
2. $2x + 3y = 9$, $5x - 4y = 11$.
3. $4a - 3b = 7$, $2a + 5b = 10$.
4. $2x - 3y = 2$, $3x - 2y = -2$.
5. $2m + 3n = 2$, $m - 2n = 8$.
6. $2a - 9b = 4$, $4a + 3b = 1$.
7. $c - 5d = 8$, $2c - 3d = 9$.
8. $3e + 2f = 1$, $e - f = 7$.
9. $4h + 2k = 10$, $3h + k = 5$.
10. $h - 2k = 8$, $2h - 4k = 12$. Graph No. 10 and explain the result.

Solve each of the following by graphing or by one of the formal methods and check the results:

11. $x + 2y = 7$, $2x - y = 4$.
12. $3x - y = 9$, $x - 2y = 8$.
13. $2x - 3y = 6$, $2x + y = 6$.
14. $5x - 2y = 4$, $3x + 2y = -4$.
15. $x = 15 - 5y$, $x = 1 + 2y$.
16. $x = 1 - y$, $y = 5 - 2x$.
17. $3x + 1 = y$, $x + y = -7$.
18. $5x = 12 - 3y$, $5y = 3x - 14$.
19. $x = 7y + 14$, $y = 7x - 2$.
20. $8x + 5y + 1 = 0$, $2y = 8 + x$.
21. $9x - 14y = 27$, $4x + 3y = 12$.
22. $7x - 5y = 10$, $5x + 7y = -14$.
23. $x - 2 = y + 3$, $y - 7 = 4 - x$.
24. $3(x - y) - 5(x + y) = 8$, $5(x - 2) - 7(y + 3) = 3$.
25. $3(y + 2) = 6 - 4(x - y)$, $5x = 8 - 4(y + 2)$.

Exercise 75. Problems

Solve the following problems using two unknowns:

1. The sum of two numbers is 30 and their difference is 6. Find the numbers.

2. Find two numbers whose sum is 8 and 5 times the first plus 3 times the second is 34.

3. Two numbers differ by 6. If the smaller is added to twice the larger, the sum is 24. What are the numbers?

4. Divide 24 into two such parts that the sum of 3 times one and 5 times the other is 100.

5. A is 3 times as old as B. Ten years ago he was 5 times as old. Find their ages now.

6. The admission to a certain foot-ball game was 35 cents for members of the school at which the game was held and 50 cents for others. 800 tickets were sold for $325.00. Find the number of tickets of each kind. (Hint. $x+y=800$, and $35x+50y=32500$). Ans. 500 and 300.

7. James has $6.75 in dimes and quarters. If he has 42 coins altogether, how many has he of each kind?

8. A grocer mixes 100 pounds of tea at a cost of $68.00 from two kinds of tea worth 80 cents and 50 cents, respectively. How many pounds of each kind did he use?

9. A chemist has an acid in two forms, one 90% pure and the other 60% pure. How much of each must he take to make a mixture of 10 ounces 72% pure?

Suggestions. Let x be the number of ounces used of the 90% and y the number of ounces of the 60%.

By the conditions of the problem $x+y=10$ (1) and

$$\frac{90x}{100}+\frac{60y}{100}=\frac{72(10)}{100}. \quad (2)$$

10. A druggist has alcohol in two strengths, 75% pure and 95% pure. How much of each must he use to obtain 8 oz. 90% pure? How much to obtain 10 oz. 85% pure?

11. A chemist has an acid in two forms such that if he takes 4 oz. of the first and 6 oz. of the second he will have an acid 85% pure, but, if he takes 4 oz. of the first and 1 oz. of the second, the result will be 75% pure. What is the per cent of purity of each form?

Hint. Let x and y be the number of per cent in the purity of each form respectively.

The equations are $\frac{4x}{100}+\frac{6y}{100}=\frac{85(10)}{100}$ and $\frac{4x}{100}+\frac{y}{100}=\frac{75(5)}{100}$

12. A druggist has two solutions of a certain drug such that 10 oz. of the first and 5 oz. of the second makes a 20% solution, while 2 oz. of the first and 6 oz. of the second makes a 30% solution. What is the per cent of purity of the two original solutions?

13. How many pints of cream testing 4% butterfat and of cream testing 20% must be taken to make 12 pints of cream testing the legal 16%.

14. A part of $8,000 is invested in a mortgage at 6% interest and the remainder in bonds that pay 5%. Find the sum invested in each if the annual income is $450.

15. If 10 lbs. of sugar and 8 lbs. of coffee cost $4.10, and at the same price 12 lbs. of sugar and 2 lbs. of coffee cost $1.88, what is the cost of each per lb.?

Note. Our number system which we use in ordinary transactions is called the **decimal system.** The value expressed by one of the figures of a number depends upon its position in the number. These separate figures of a number are called **digits.** In the number 275, what change is produced by changing 5 to 6? By changing 7 to 8? By changing 2 to 3? By interchanging 2 and 7?

16. A number is composed of two digits. The unit's digit is 2 more than the ten's digit. Find the number if it is equal to 4 times the sum of its digits.

Solution. Let x = the ten's digit,
and y = the unit's digit.
Then $10x + y$ = the number. (Explain.)
Then $10x + y = 4(x + y)$ and $y = x + 2$.
Solving $x = 2$, $y = 4$, and $10x + y = 24$, the number.

17. The unit's digit of a number of two digits is 2 more than the ten's digit. The number is 4 less than 5 times the sum of the digits. Find the number.

18. The sum of the two digits of a certain number is 12. If 18 is added to the number, the order of the digits is reversed. What is the number?

19. The digit in unit's place is 2 less than twice the digit in 10's place. The number is 2 less than 8 times the digit in unit's place. What is the number?

20. A number is composed of three digits whose sum is 9. The middle digit is one-half the sum of the other two and the whole number is 117 times the digit in hundred's place. What is the number?

21. If a certain number of two digits is divided by the sum of the digits, the quotient is 4; and if 18 is added to the number, the order of the digits is reversed. Find the number.

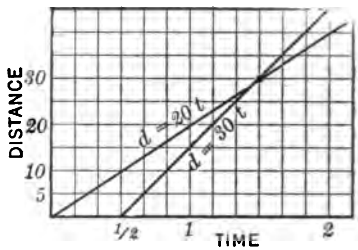
22. If a two digit number is divided by the sum of its digits, the quotient is 7. The unit's digit is 3 less than the ten's digit. What is the number?

Solve the following problems by graphing:

23. At 6 A. M. the thermometer registered 60°. It rose 2° per hour till 11 A. M., then rose 3° per hour till 4 P. M., and then fell 1° each hour till midnight. Show by a broken line on the cross section paper the changes in temperature for the 18 hours.

Suggestion. Use one horizontal space for each hour and let one vertical space represent 2°. Start with the base line as 60°.

24. Two trains start one-half hour apart from the same station and in the same direction. The first is running at the rate of 20 miles per hour, and the second at the rate of 30 miles per hour. Show by the graph when the second train will overtake the first.

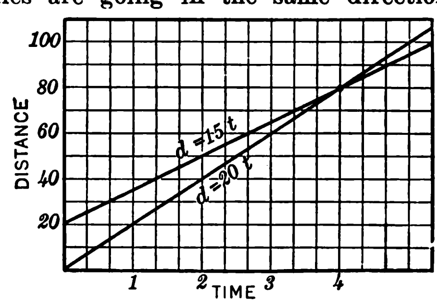


Solution. Notice in the accompanying figure that each small horizontal space represents 10 minutes, and each vertical space represents 5 miles. Explain the graph, and find the time and distance represented at the point of intersection.

25. A regiment of soldiers starts from camp, marching at the rate of 3 miles per hour. Two hours later a messenger follows them from the same camp at the rate of 5 miles per hour. How many miles will the messenger travel before he overtakes the regiment?

Hint. Care is necessary in selecting reasonable values for the horizontal, or time spaces, and for the vertical, or distance spaces.

26. Two automobiles are going in the same direction and are 20 miles apart. The one ahead is making 15 miles per hour, and the second one, 20 miles per hour. In how many hours will they be together.



Suggestion. Study the accompanying figure and explain. Rework the exercise with each horizontal space representing 5 minutes, and each vertical space representing 10 miles.

27. Three automobiles are traveling in the same direction and on the same road. The first is 12 miles ahead of the second and the second is 8 miles ahead of the third. The first is making 15 miles per hour and the second 18 miles per hour. The third overtakes the first two when they are together. Find by the graph in how many hours they will be together and the rate of the third.

28. Two railway stations, A and B, are 300 miles apart. A train starts from A toward B at the rate of 30 miles per hour, and at the same time another train leaves B for A at 45 miles per hour. In how many hours will they meet?

Note. The point B is directly above A on the cross section paper. One line of the graph begins at B and slopes downward.

29. A man travels from A to B on a train making 30 miles per hour and immediately returns on another train making 50 miles per hour. If the whole trip requires 8 hours, find the distance from A to B.

Suggestion. The graph representing the return trip may be discovered by determining how far he is from A in 7 hours, 6 hours, etc. This second line of the graph slopes upward and to the left.

30. Two boys are saving their money to invest in War Savings Stamps. On January 1st, one has $3.00 and is saving 50 cents per week, and the other has $1.50 and is saving 75 cents per week. In how many weeks will they have the same amount?

31. On January 1st, Henry begins to save his money and saves 80 cents each week. Four weeks later James begins to save at the rate of $1.00 per week. When will they have equal amounts?

32. Tom was 48 inches tall at 10 years of age. His average rate of growth was 3 inches per year till he was 14 years of age, and then 4 inches per year till he was 16 years of age. Represent his growth by a graph.

38. Graph the equation $5F = 9C + 160$ and by the use of the graph determine the Centigrade reading when the Fahrenheit thermometer registers $5°$; $50°$; $70°$.

89. Simultaneous systems involving fractions.

Exercise 76

Clear each equation of fractions and solve, checking in the original equations:

1. $\dfrac{x}{2}+\dfrac{y}{3}=6$, $\quad \dfrac{x}{3}-\dfrac{y}{9}=1.$

2. $\dfrac{3a}{4}-\dfrac{b}{5}=2$, $\quad \dfrac{a}{2}+b=7.$

3. $\dfrac{x+1}{2}+y=10$, $\quad 2x-y=3.$

4. $y=\dfrac{x-11}{2}$, $\quad x=\dfrac{y-7}{3}.$

5. $\dfrac{m-n}{2}=3(m+n)$, $\quad \dfrac{m+1}{4}+\dfrac{n+1}{2}=0.$

6. $\dfrac{x-1}{2}+\dfrac{y-2}{2}=1$, $\quad \dfrac{2x-1}{3}+\dfrac{3y-1}{2}=5.$

7. $\dfrac{3x+2}{4}+\dfrac{2y}{3}=4$, $\quad \dfrac{x+4}{3}-\dfrac{2y}{6}=1.$

8. $\dfrac{2x+3}{2}+\dfrac{x+y}{8}=5$, $\quad \dfrac{3x-1}{4}+\dfrac{5y+1}{3}=4.$

9. $\dfrac{a-b}{5}+\dfrac{a+2b}{5}=\dfrac{6}{5}$, $\quad \dfrac{2a+b}{2}-\dfrac{a-2b}{3}=\dfrac{1}{3}.$

10. $\dfrac{x-y}{2}-\dfrac{x-4y}{3}=-1$, $\quad \dfrac{x+1}{5}-\dfrac{y-4}{3}=3.$

11. $\dfrac{x+3}{2}+5y=9$, $\quad \dfrac{y+9}{10}-\dfrac{x-2}{3}=0.$

12. $\dfrac{3R+5r}{10}-\dfrac{R+r}{4}=1$, $\quad \dfrac{R+3r}{7}+\dfrac{2R+5r}{5}=7.$

13. $\dfrac{3x}{4}+\dfrac{4y}{6}=\dfrac{7}{12}$, $\quad \dfrac{x+1}{2}-\dfrac{y+1}{3}=\dfrac{1}{6}.$

14. $\dfrac{x+y}{4}-\dfrac{x-y}{2}=3$
$\dfrac{2x+3y}{7}+\dfrac{x+y}{4}=5.$

15. $\dfrac{2x}{3}-\dfrac{3y}{8}=7$
$\dfrac{y}{4}-\dfrac{y-x}{7}=0.$

Some systems of simultaneous equations involving the unknowns in the denominators may be solved by eliminating one of the unknowns without clearing of fractions, as follows:

16. $\dfrac{1}{x}-\dfrac{1}{y}=1$ (1)
$\dfrac{1}{x}+\dfrac{1}{y}=5$ (2).

Adding (1) and (2) gives $\dfrac{2}{x}=6$ and $x=\dfrac{1}{3}$. Explain. Solve for y and check.

17. $\dfrac{2}{3x}+\dfrac{3}{4y}=\dfrac{11}{2}.$ (1)
$\dfrac{3}{4x}-\dfrac{2}{y}=\dfrac{1}{2}.$ (2).

Solution. $\dfrac{1}{x}+\dfrac{9}{8y}=\dfrac{33}{4}.$ (3) Multiplying (1) by $\dfrac{3}{2}$.

$\dfrac{1}{x}-\dfrac{8}{3y}=\dfrac{2}{3}.$ (4) Multiplying (2) by $\dfrac{4}{3}$.

$\dfrac{9}{8y}+\dfrac{8}{3y}=\dfrac{33}{4}-\dfrac{2}{3}.$ Subtracting (4) from (3).

Whence $27+64=198y-16y$, and $y=\frac{1}{2}$.
Solve for x and check.

18. $\dfrac{2}{x}+\dfrac{1}{y}=4$ (1)
$\dfrac{1}{x}+\dfrac{3}{y}=5$ (2).

Suggestion. Multiply (2) by 2 and subtract from (1).

19. $\dfrac{3}{x}+\dfrac{3}{2y}=9$
$\dfrac{2}{x}-\dfrac{3}{y}=-6.$

20. $\dfrac{1}{2x}+\dfrac{1}{3y}=4$
$\dfrac{1}{3x}-\dfrac{1}{9y}=\dfrac{2}{3}.$

21. $\dfrac{1}{x}+\dfrac{1}{y}=7$
$\dfrac{3}{4x}+\dfrac{2}{3y}=5.$

90. Simultaneous equations of three variables. A system of simultaneous equations of more than two variables can be solved by the method of addition or subtraction. Such a system must contain as many equations as there are unknowns. The complete process is shown in the following **illustrative example:**

$$2x + 3y + 2z = 7 \quad (1)$$
$$4x - 2y - 3z = 1 \quad (2)$$
$$5x - 5y - 7z = -6 \quad (3)$$

Multiplying (1) by 2, $4x + 6y + 4z = 14$

Multiplying (2) by 3, $12x - 6y - 9z = 3$

Adding, $16x - 5z = 17 \quad (4)$

Multiplying (2) by 5, $20x - 10y - 15z = 5$

Multiplying (3) by 2, $10x - 10y - 14z = -12$

Subtracting, $10x - z = 17 \quad (5)$

Multiplying (5) by 5, $50x - 5z = 85$

$16x - 5z = 17$

Subtracting, $34x = 68$ Whence $x = 2$.

Substituting $x = 2$ in (5), $20 - z = 17$, whence $z = 3$.

Substituting $x = 2$ and $z = 3$ in (1), $4 + 3y + 6 = 7$, $y = -1$.

Checking, $4 - 3 + 6 = 7$, $8 + 2 - 9 = 1$, $10 + 5 - 21 = -6$.

Exercise 77

Solve the following simultaneous systems checking in all three equations:

1. $x + 3y + 2z = 13$
   $3x + 2y + z = 10$
   $2x + y + 3z = 13.$

2. $4a + 2b - 4c = -4$
   $5a + b - 3c = 14$
   $3a - 5b + 2c = 6.$

3. $2a + 3b + 4c = 29$
   $a + b + 2c = 13$
   $3a + 2b + c = 16.$

4. $2m - n + 4t = 24$
   $m - 2n + 2t = 18$
   $2m - 3n - 2t = 5.$

5. $a + b + c = 3$
   $a + 2b + 3c = 10$
   $2a + 2b + c = -5.$

6. $2h + 3t + 4f = 3$
   $4h - 3t + 12f = 4$
   $h + 6t + 2f = 3.$

7. $2x + 3y - 5z = -16$
$3x - 2y - 3z = -3$
$5x + y + 2z = 6.$

8. $3a + 2b - c = 10$
$2a - b + 3c = -7$
$a + 7b - 5c = 14.$

9. $3x + 5y + 4z = -11$
$4x - 2y - 7z = -4$
$2x + 3y - 6z = 4.$

10. $2x + 3y - 3z = -21$
$5x - 2y - 5z = 3$
$2x + 5y = -8.$

11. $3z - x - y = 28$
$5y + z - 2x = -7$
$2z - 3y = 26.$

Suggestion. Eliminate z from the first two equations, and combine the resulting equation with the third.

12. $a + b = 8$
$b + c = 12$
$a + c = 10.$

13. $x + 2y = 5$
$y + 2z = 8$
$z + 2x = 5.$

14. $x + y = 3$
$y + z = 5$
$z + w = 7$
$w - x = 3.$

15. $m + n + r = 9$
$m - n = 2$
$n - r = 1.$

16. $x + y + z + w = 10$
$x + 2y - 3z + 4w = 8$
$x - z = 2$
$y - 2w = 1.$

Eliminate before clearing of fractions in these:

17. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 12$
$\frac{1}{x} + \frac{2}{y} - \frac{1}{z} = 6$
$\frac{2}{x} - \frac{3}{y} + \frac{2}{z} = 4.$

18. $\frac{1}{x} - \frac{1}{y} = \frac{1}{6}$
$\frac{1}{y} - \frac{1}{z} = \frac{1}{12}$
$\frac{1}{x} + \frac{1}{z} = \frac{3}{4}.$

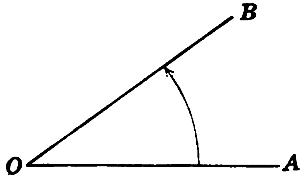
19. $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{3}{2}$
$\frac{1}{x} + \frac{1}{y} = \frac{5}{12}$
$\frac{1}{y} + \frac{1}{z} = \frac{7}{24}.$

20. $\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = \frac{5}{3}$
$\frac{2}{x} + \frac{3}{y} + \frac{1}{z} = \frac{7}{6}$
$\frac{3}{x} + \frac{1}{y} - \frac{2}{z} = -\frac{7}{6}.$

PROBLEMS FROM GEOMETRY

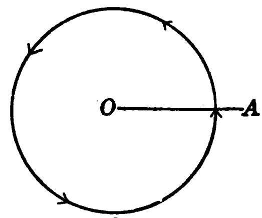
91. Angles and their problems. If two straight lines extend from a point in different directions, they are said to form an **angle.** The point is called the **vertex** and the lines the **sides** of the **angle.**

The **size** of an angle depends upon how much the lines **differ** in **direction** and it is measured by the amount that it is necessary to turn one of the sides about the vertex as a center to bring it into the position of the other side. See figure 1.



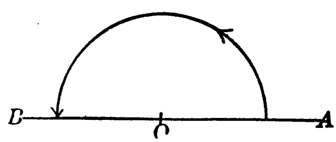
1.

If a line that extends in one direction from a point is swung about the point (see figure 2) until it has made one complete turn about the point and arrived at its original position, it is said to have turned through an angle of 360°, or a **perigon.**



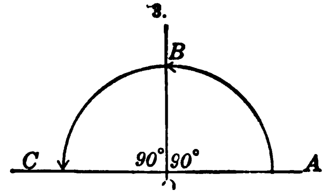
2.

If it has made one-half of such a revolution (see figure 3), it has turned through an angle of 180°, or a **straight angle**—so called because the two sides extend in opposite directions from the vertex forming a straight line.



3.

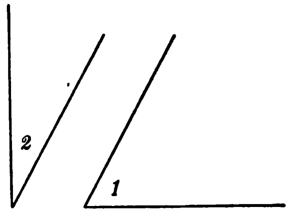
If the moving line swings through one-fourth of a complete revolution or 90°, a **right angle** is formed. If two lines meet at right angles, they are said to be **perpendicular** to one another. See figure 4.



4.

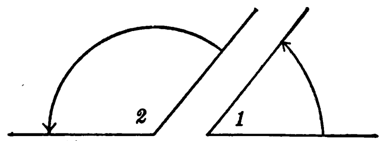
The right angle is a convenient unit of measure.

If two angles are of such size that their sum is a right angle, they are called **complementary angles.** One is called the **complement** of the other.



COMPLEMENTARY ANGLES

If two angles are such that their sum is a straight angle, they are called **supplementary angles.** One is called the **supplement** of the other.



SUPPLEMENTARY ANGLES

An angle less than a right angle is called an **acute angle.**

Exercise 78

1. Two angles are supplementary and one is twice the other. Find the number of degrees in each angle.

Suggestion. Let x be the number of degrees in the smaller angle, then $2x$ will be the number of degrees in the larger angle and $x + 2x = 180$. Solve and check.

2. Two angles are complementary and one is 10 degrees more than the other. Find each angle.

3. Two angles are complementary and one-half of one equals one-seventh of the other. Find each angle.

4. What angle is four times its complement? three times?

5. Two angles are supplementary and one is 30 degrees less than the other. Find the number of degrees in each angle.

6. Two angles are complementary and the larger is the supplement of four times the smaller. Find the angles.

7. Two angles are complementary. If the first is doubled and the second increased by 50 degrees, the resulting angles are supplementary. Find the angles.

8. Find two complementary angles whose sum is 3 times their difference.

9. Find the angle one-half of whose complement equals one-fifth of its supplement.

10. There are three angles whose sum is 360 degrees. The first is the supplement of one-fifth of the second and the sum of the first two exceeds the third by 80 degrees. Find the angles.

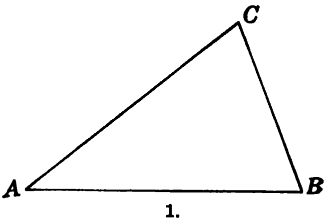
11. Three radii are drawn in a circle forming equal angles at the center. Find the value of each angle.

12. Four radii are drawn in a circle forming four angles at the center. The second of these angles is twice the first and the third is twice the second; the fourth exceeds the first by 40 degrees. Find the value of each angle.

13. Four radii are drawn in a circle. Find the value of each of the four angles if the first and second are supplementary, the first and fourth complementary, and the third is 80 degrees more than the fourth.

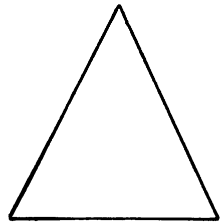
14. Find two angles whose sum is 130 degrees if the complement of one is one-third the supplement of the other.

92. Triangles and their problems. If three points not in a straight line are connected by straight lines, the figure formed is called a **triangle**. The points are called the **vertices** and the lines the **sides** of the triangle. If no sides are equal, the triangle is called **scalene** (Fig. 1). If two sides are equal, it is



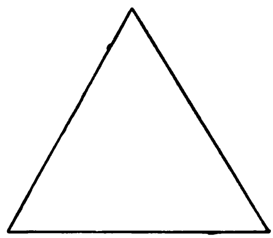
1.

called **isosceles** (Fig. 2), and if all three sides are equal, it is **equilateral**. (Fig. 3).


ISOSCELES
2.

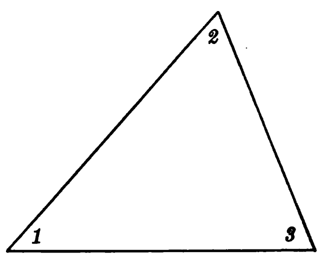
Evidently a triangle has three angles, one at each vertex.

It is proved in geometry that the sum of the three angles of a triangle is two right angles or 180 degrees.


EQUILATERAL
3.

The truth of this theorem may be tested by cutting a triangle out of paper, then tearing off the corners, or angles, and arranging them as in figure 4. Evidently their sum is approximately a straight angle.

If one of the angles of a triangle is a right angle, the triangle is called a **right triangle**.

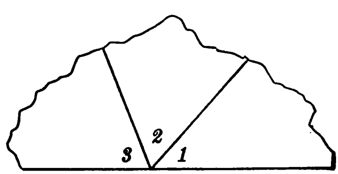


The longest side, that opposite the right angle, in a right triangle is called the **hypotenuse**.

Ex. 1. Draw an isosceles right triangle.

Ex. 2. Draw a scalene right triangle.

Ex. 3. Show that the acute angles of a right triangle are complementary.



4.

Exercise 79

1. How many degrees in each angle of a triangle whose angles are equal?

2. The two acute angles of a right triangle are equal. Find each angle.

3. Two of the angles of a triangle are complementary and one of the two is twice the other. Find the angles.

4. The second of the three angles of a triangle is 10 degrees more than the first and the third is 40 degrees more than the second. Find the angles.

5. How many degrees are there in the angles of a triangle if the second is twice the first and the third is 20 degrees less than the second?

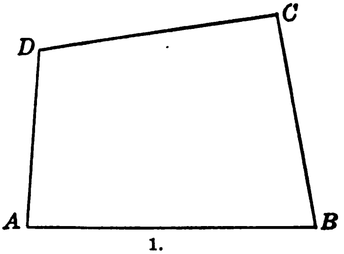
6. Find the angles of a right triangle if one of the two acute angles is 50 degrees more than the other.

7. The number of degrees in the angles of a triangle are x, $x + 20$, and $3(x - 15)$. Find x and each angle.

8. The numbers of degrees in the angles of a triangle are $3x$, $x + 35$, and $x^2 + 5$. Find x and the angles.

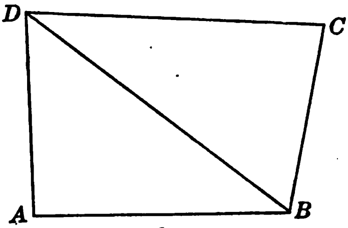
9. Two of the sides of a triangle are equal and the third side is 6 inches less than the sum of the other two. Find the sides if their sum is 46 inches.

93. The quadrilateral and its problems. If four points, no three in the same straight line, are arranged in consecutive order and connected by straight lines as A, B, C, and D, in Fig. 1, the figure formed is called a **quadrilateral.**



1.

A quadrilateral has four sides and four angles.



2.

If two opposite vertices are connected by a straight line as in Fig. 2, the quadrilateral is divided into two triangles, and since the sum of the angles of each triangle is two right angles, the sum of the angles of the quadrilateral is four right angles.

A quadrilateral each of whose angles is a right angle is called a **rectangle.**

Exercise 80

1. The four angles of a quadrilateral are such that the second is twice the first, the third twice the second, and the fourth twice the first. Find all four angles.

2. Find the angles of a quadrilateral if the first and second angles are supplementary, the second and third complementary, and the fourth five times the third.

3. Find the angles of a quadrilateral if the second is 30 degrees more than the first, the third 30 degrees more than the second, and the fourth three times the first.

4. Find the angles of a quadrilateral if two opposite angles are supplementary and equal and the other two differ by 40 degrees.

5. Find the sides of a quadrilateral if the first is 1 inch longer than the second, the second 1 inch longer than the third, the third 1 inch longer than the fourth, and the sum of all four sides 34 inches.

6. A certain quadrilateral has three equal sides and the fourth side is equal to half the sum of the other three. Find each side if their sum is 27 inches.

SYSTEMS PARTIALLY QUADRATIC

94. Simultaneous equations, one linear and one quadratic. If a system of two equations in the same two unknowns has one equation of the first degree and one of the second, the system may be solved by finding from the first degree equation the value of one of the unknowns in terms of the other and substituting this value in the other given equation.

Example 1. Given $xy = 18$ (1)
and $x + y = 11$ (2).
From (2) $x = 11 - y$.
Substituting this value of x in (1)
gives $(11 - y)y = 18$, or $11y - y^2 = 18$.
Therefore $y^2 - 11y + 18 = 0$.
Factoring, $(y - 9)(y - 2) = 0$. Therefore $y = 9$, or 2.
Substituting these values of y in (2), $x = 2$, or 9.

Note. It must be kept in mind that these values are in pairs. That is, when $y = 9$, $x = 2$, and when $y = 2$, $x = 9$.

Check both sets of values in both equations.

Example 2. $x^2 - 3y^2 = 13$ (1).
$x - 2y = 1$ (2).
From (2), $x = 2y + 1$.
Substituting in (1), $(2y + 1)^2 - 3y^2 = 13$.
Whence $4y^2 + 4y + 1 - 3y^2 = 13$, $y^2 + 4y - 12 = 0$.
Solving, $y = 2$, or -6, and $x = 5$, or -11. Check.

Exercise 81

Solve each of the following systems:

1. $x + y = 7$, $x^2 + y^2 = 25$.
2. $x - y = 5$, $xy = 14$.
3. $x + y = 5$, $x^2 - 2y^2 = 1$.
4. $xy = 18$, $x - y = 3$.
5. $x^2 + y^2 = 50$, $x - y = 6$.
6. $x + 2y = 20$, $xy = 48$.
7. $x^2 + y = 39$, $x + y = 9$.
8. $x^2 + xy + y^2 = 19$, $2x - 3y = 0$.

9. $x^2 - xy + y^2 = 13$
$x - y = 1.$

10. $4x^2 - 9y^2 = 0$
$x + y = 10.$

11. $x^2 + xy + y^2 = 49$
$x - y = 2.$

12. $x^2 + xy = 28$
$x + y = 4.$

13. The sum of two numbers is 9 and the sum of their squares is 41. Find the numbers.

14. The difference of two numbers is 2 and the difference of their squares is 28. What are the numbers?

15. The sum of two numbers is 1 and the sum of their squares is 1. What are the numbers?

16. Find two numbers whose sum is 3 and the sum of whose squares is 29.

17. Find two numbers whose difference is 11 and the sum of whose squares is 61.

18. A rectangle is 7 inches longer than wide and its area is 120 square inches. Find its dimensions.

19. The sum of the length and width of a rectangle is 14 inches and the area is 45 square inches. Find its dimensions.

20. Find the dimensions of a rectangle whose perimeter is 20 inches and whose area is 21 square inches.

21. The area of a triangle is 54 square inches. Its base is 3 inches more than its altitude. Find both.

22. Divide the number 12 into two parts, the sum of whose squares shall be 74.

23. Find two numbers whose difference is 3, and whose sum multiplied by the smaller is 65.

24. Divide the number 20 into two parts such that one part is equal to the square of the other part.

25. Find the price of apples per bushel if the number

of bushels that can be bought for $10 is 3 more than the number of dollars per bushel.

26. The denominator of a certain fraction is 5 more than the numerator. If the numerator be decreased by 1, and the denominator be decreased by 7, the product of the resulting fraction and the original fraction is $\frac{3}{4}$. Find the fraction.

27. The sum of two numbers is equal to 6 times their difference, and the difference of their squares is 24. Find the numbers.

28. A certain rectangle is 2 feet longer than wide. If its width were increased by 2 feet, and its length were increased by 3 feet, the area would be doubled. Find the dimensions of the rectangle.

29. The sum of the length and width of a rectangle is 15 feet. If the width were increased 1 foot, and the length were increased 2 feet, the area would be 80 square feet. Find the dimensions.

30. The difference of two numbers is 4. The square of their sum exceeds the sum of their squares by 24. Find the numbers.

31. The difference of two numbers is 2. The square of their sum exceeds the difference of their squares by 48. Find the numbers.

32. John takes a long walk into the country. The number of miles he walks per hour is 2 less than the number of hours he walks. If the whole journey is 15 miles, find the number of hours he walks and the rate per hour.

33. John paid 85 cents for some oranges. He discovers that he has 12 more oranges than the number of cents he paid per orange. What was the price of the oranges and how many oranges did he buy?

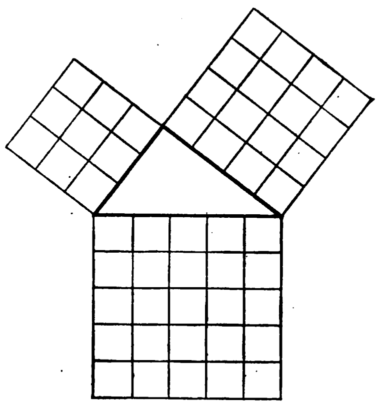
95. Pythagorean Proposition. One of the famous theorems of geometry is known as the **Pythagorean Proposition,** so called because its first proof is accredited to Pythagoras, a Greek mathematician who lived and taught about 550 B. C.

The theorem is stated as follows:

The square on the hypotenuse of a right triangle is equal to the sum of the squares on the two sides about the right angle.

(For definition of hypotenuse see § 92.)

While there are many known proofs of this theorem, several of which will be met in the study of geometry, the truth becomes apparent from the study of the accompanying figure.



Let the student cut squares from cross section paper, whose sides are respectively 6, 8, and 10 units and place them so their sides enclose a triangle.

Similarly, try 3, 4, and 5 units; 5, 12, and 13 units; and 8, 15, and 17 units.

These are particular right triangles, all of whose sides are integers.

Can you think of other right triangles all of whose sides are integers?

Can a right triangle have its sides 5, 6, and 8, units?

Let the student look up and report on the history of Pythagoras and of this theorem.

Exercise 82

1. The hypotenuse of a right triangle is 13 inches and the sum of the other two sides 17 inches. Find the sides.

Suggestion. Let x and y be the lengths of the two sides about the right angle.

Then $x^2 + y^2 = 13^2$ from the theorem,

and $x + y = 17$ from the conditions of the problem.

2. The difference between the two sides of a right triangle is 7 inches and the hypotenuse is 17 inches. Find the sides.

3. The hypotenuse of a right triangle is 25 inches and the difference of the other two sides is 5 inches. Find the sides.

4. The sum of the two sides of a right triangle is 35 inches and its area is 150 square inches. Find the sides and the hypotenuse.

Suggestion. The area is one-half the product of the two sides.

5. The hypotenuse of a right triangle is 8 inches longer than one of the sides and 4 inches longer than the other Find the hypotenuse and the sides.

Suggestion. Let the hypotenuse be x inches long and the sides $x - 8$ and $x - 4$ inches, respectively

Then $x^2 = (x - 8)^2 + (x - 4)^2$

6. The hypotenuse of a right triangle is 10 inches longer than one of the sides and 20 inches longer than the other. Find the hypotenuse and the sides,

7. The perimeter (sum of the two sides and the hypotenuse) of a right triangle is 30 inches. The lengths in inches of the longer side and the hypotenuse are consecu tive numbers. Find the sides and the hypotenuse.

Suggestion. If the longer side is represented by x, what is the hypotenuse? Let y represent the shorter side.

Then $(x + 1)^2 = x^2 + y^2$ and $x + x + 1 + y = 30$.

8. The perimeter of a right triangle is 56 inches and the numbers of inches in the longer side and the hypotenuse are consecutive numbers. Find the sides and the hypotenuse.

9. The numbers of inches in the sides and the hypotenuse of a right triangle are three consecutive numbers. Find them.

10. The perimeter of a rectangle is 28 inches and the diagonal is 10 inches. Find the length and width.

Suggestion. Draw the figure and notice that the diagonal of a rectangle divides it into two right triangles. Keep in mind that the perimeter is the sum of twice the width and twice the length.

11. The perimeter of a rectangle is 14 inches and the diagonal is 5 inches. Find the length and width.

12. A trunk is 7 inches longer than it is wide and the longest cane that can lie on the bottom is 35 inches. Find the width and length of the trunk.

13. One side of a right triangle is 12 inches and the hypotenuse is 3 inches less than twice the other side. Find the hypotenuse and the side.

14. The perimeter of a right triangle is 30 inches and one side is 12 inches. Find the hypotenuse and the other side.

15. The hypotenuse of a right triangle exceeds one of the sides by 3 inches and the other by 6 inches. Find the hypotenuse and the sides.

16. The perimeter of a right triangle is 60 inches and the numbers of inches in the longer side and the hypotenuse are consecutive even numbers. Find the sides and the hypotenuse.

CHAPTER VII

FORMULAS AND LITERAL EQUATIONS

96. Place and importance of formulas. The student will recall that a number of rules in arithmetic were conveniently stated in the compact form known as a formula. Among these were the following:

(1) The area of a circle was expressed as, $A = \pi r^2$.

(2) The area of a rectangle was expressed as, $A = lw$.

(3) The interest on a note was expressed as, $i = prt$.

A **formula** is evidently an equation of condition, so arranged that one literal number depends for its value upon, or is a **function** of, one or more other literal numbers.

The high school texts in physics contain many formulas, some of which will be met with in the paragraphs that follow.

97. Solution of a formula for different literal numbers. In an equation of the form $A = lw$, where one literal number is a function of several others, it is necessary that the numerical value of each of the literal numbers in the right member of the equation be known.

If the numerical values of A and l are known and w is required, the equation $A = lw$ may be transformed into the equation $w = \frac{A}{l}$ by dividing both sides of the original equation by l. Similarly, show how $l = \frac{A}{w}$ may be obtained.

The **transformation** of the equation $A = lw$ into the equations $l = \frac{A}{w}$ and $w = \frac{A}{l}$ is known as the **solution** of the equation for l and w.

Exercise 83

State the necessary formula and solve the following:

1. The area of a field is 1200 square rods and its width is 30 rods. What is its length?

2. What is the width of a board 6 feet long, if it contains 4 board (or square) feet?

Note. A board foot for all thicknesses of lumber one inch or less is a board whose length times its width gives a product of 144 square inches, or one square foot.

Let x be the width of the board in feet.

Then $6x = 4$, and $x = \frac{2}{3}$. Therefore the board is $\frac{2}{3}$ of a foot or 8 inches in width.

3. Five boards of the same width and each 12 feet long are sold as 40 board feet. What is their width?

4. What is the length of a board 9 inches wide if it contains 12 board feet?

5. What is the width of a board 12 feet long if it is sold as 10 board feet?

Note. Timbers are sold by the number of board feet that they contain. This is determined by the formula $A = \frac{lwt}{12}$ where l is the length in feet, and w and t are the width and thickness in inches.

A square timber 8 inches by 8 inches and 18 feet long contains $\frac{8 \cdot 8 \cdot 18}{12}$ or 96 board feet.

6. How long must a 4 by 6 post be to contain 32 board feet?

7. What will 40 2 by 8's each 16 feet long cost at \$60 per M, or 1000 board feet?

8. How many 2 by 8's, each 16 feet long, can be purchased for \$51.20, if this lumber costs \$80.00 per M?

9. How many 2 by 4's, 12 feet long, will equal one M?

98. The interest formulas.

Solving the interest formula $i = prt$ for t gives $t = \frac{i}{pr}$.

Explain how this result is obtained.

Similarly solve for p and r.

Exercise 84

State the formula and solve:

1. If the interest on a note for 2 years and 6 months at 5% is \$150.00, what is the principal, or face, of the note?

2. In what time will \$1,600 yield \$120 at 6%?

3. At what rate will \$2,000 yield \$140 interest in 1 year and 9 months?

4. The amount of a note is found by adding the interest to the face of the note. The formula is $a = p + prt$.

Solve this formula for r. Ans. $r = \frac{a - p}{pt}$.

Solve also for t and p.

Note. In solving for p it is necessary to get all the terms that contain p into the left member, then factor this member into two factors, one of which is p.

The steps of the solution are as follows:

$p + prt = a$, $p(1 + rt) = a$,

and $p = \frac{a}{1 + rt}$. Explain fully.

5. At what rate will \$2,000 amount to \$2,200 in 2 years?

Using the formula $r = \frac{a - p}{pt}$, $r = \frac{2200 - 2000}{2000 \times 2} = \frac{200}{4000} = \frac{5}{100}$.

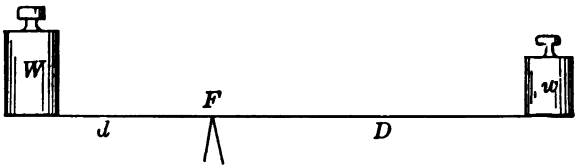
$\frac{5}{100}$ is the same as 5%.

6. In what time will \$3,000 amount to \$3,450 at 5%?

7. What principal will amount to \$5,400 in 2 years and 6 months at 5%?

8. What will \$2,500 amount to in 18 months at 6%?

99. The formula arising from the law of levers.



If two unequal weights are placed on the ends of a bar, or **lever,** and the bar is balanced, as in the accompanying figure, at the point F called the **fulcrum,** it will be found that the weight times its distance from the fulcrum on one side just equals the weight on the other side times its distance from the fulcrum.

This is know as the **law of levers.**

Its formula is $Wd = wD$.

Note. W and w are the numbers of units of weight and D and d the numbers of units of distance.

Let the student solve this for each of the four numbers W, d, D, and w.

Exercise 85

1. Two boys whose weights are 100 and 120 pounds respectively, find that they balance when playing at teeter when the smaller boy is just 6 feet from the fulcrum. How far is the larger boy from the fulcrum?

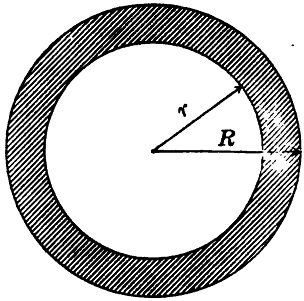
2. Two boys balance at teeter when they are 6 feet and $7\frac{1}{2}$ feet, respectively, from the fulcrum. If the smaller boy weighs 100 pounds, what is the weight of the larger?

3. What would be the weight of the smaller boy in No. 2 if the larger boy weighed 100 pounds?

4. How far from the fulcrum on the opposite side must a weight of 20 pounds be placed to balance a weight of 30 pounds 3 inches from the fulcrum?

5. If W is twice w, D must be what multiple of d?

100. Area between concentric circles.



If two circles are concentric (have a common center), the area of the ring between their circumferences is found by taking the difference between their areas.

This gives the formula, $A = \pi R^2 - \pi r^2$, which may be factored as follows:

$$A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi(R - r)(R + r).$$

Assume that $\pi = \frac{22}{7}$, unless otherwise indicated.

Exercise 86

1. Find the area of the ring between two concentric circles whose radii are 10 inches and 4 inches.

Solution. $A = \frac{22}{7}(10 - 4)(10 + 4) = 264$. Ans. 264 sq. in.

2. What is the area of the ring between two concentric circles whose radii are 12 and 9 inches, respectively?

3. What is the area of the ring between two concentric circles whose radii are 15 and 6 inches, respectively?

4. The difference between the radii of two concentric circles is 2 inches and the area of the ring between their circumferences is 88 square inches. What are their radii?

5. The area between two concentric circumferences is 1848 square inches and the radius of the larger is 14 inches more than that of the smaller. Find their radii.

6. A circular flower bed whose radius is 19 feet is surrounded by a border 5 feet wide. Find the area of the border.

7. At 20 cents per square foot what will it cost to build a walk 3 feet wide around the bed and border of No. 6?

101. Miscellaneous formulas.

Exercise 87

1. How high is a cliff if a stone dropped from its top reaches the ground in 3 seconds?

If an object is dropped from some elevated point and its fall toward the earth is not interrupted, the number of feet, d, through which it will fall in t seconds is given by the formula $d = 16t^2$.

2. A bomb dropped from an aeroplane reaches the ground in 10 seconds. How high is the plane?

3. In time, rate, and distance problems, $d = tr$. What is the formula for t in terms of d and r? For r in terms of d and t?

4. How long will it take an automobile to go 90 miles if its speed is 15 miles per hour? What formula is used?

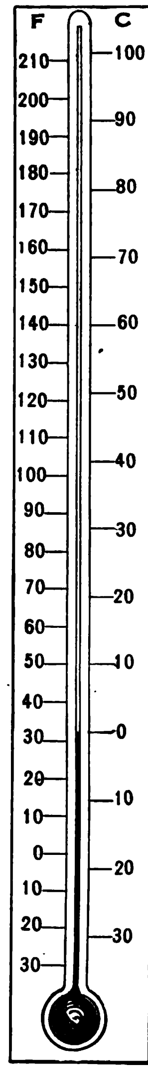
5. What is the rate of an automobile if it goes 90 miles in 3 hours and 30 minutes?

6. The horse power of a steam engine is given by the formula $h = \frac{plan}{33000}$, where p is the steam pressure in pounds per sq. in., l is the length of the stroke in feet, a is the area of the piston in square inches, and n is the number of strokes per minute.

Solve the formula for l in terms of the other numbers. For n.

7. What is the horse power of an engine if $p = 60$, $l = 2$ (feet), a is the area of a circle whose radius is 8 inches, and $n = 100$?

8. A marine engine develops 32 horse power with a piston that has an area of $314\frac{2}{7}$ square inches, length of stroke 28 inches, and 60 pounds pressure per square inch. Find n.



9. If in No. 8 the length of the stroke were 21 inches instead of 28 inches, how many horse power would the engine develop?

10. The accompanying cut of a thermometer has the Centigrade scale on the right and the Fahrenheit scale on the left of the mercury column. Notice that the freezing point of water is 0° Centigrade (C) and the boiling point is 100°, while these points are 32° and 212° on the Fahrenheit (F) scale.

Can you explain why the ratio of 1° C to 1° F is 9 : 5?

The formula for changing a Centigrade reading into a Fahrenheit reading is $F = 32 + \frac{9}{5}C$, where F and C are the numbers of degrees in the readings on the two scales for the same temperature.

Explain the formula and solve for C.

11. 50° Fahrenheit is the same as what reading Centigrade? 0° Fahrenheit is the same as what reading Centigrade?

Check both readings from the figure.

12. Translate 60° Centigrade into degrees Fahrenheit. $-40°$ Centigrade is the equivalent of what reading Fahrenheit?

13. 1000° Centigrade equals how many degrees Fahrenheit? 1000° Fahrenheit equals how many degrees Centigrade?

14. Check each multiple of 10° on each scale to determine how accurately the engraver has drawn the Centigrade scale with reference to the Fahrenheit scale.

102. Literal equations. Every formula that has been presented in this chapter is an equation in which the value of a single literal number is expressed in terms of other literal numbers.

An equation, the solution of which for any particular literal number gives an answer in terms of one or more other literal numbers, is called a **literal equation.**

Such equations may be solved for any one of the literal numbers, but if letters near the end of the alphabet such as x, y, and z are used to represent numbers, together with letters from the beginning of the alphabet such as a, b, and c, the literal numbers a, b, c, etc., are commonly considered as **known numbers,** and x, y, and z as **unknown.**

Unless otherwise specified, the equation is solved when the value of the unknown number is found.

Exercise 88

Solve each of the following and check:

1. $\frac{x}{c} = a$. Clearing of fractions gives $x = ac$. Ans.

2. $\frac{bx}{a} = b^2$. Show $bx = ab^2$. $x = ab$. Ans.

3. $ax + b = c$. Ans. $x = \frac{c - b}{a}$.

4. $ax + b^2 = bx + a^2$.

Solution. (1) $ax - bx = a^2 - b^2$.
(2) $x(a - b) = (a - b)\ (a + b)$,
(3) and $x = a + b$. Explain and check.

5. $ax - bx = a^2 - 2ab + b^2$. Ans. $x = a - b$.

6. $4ax + 1 = ax + 7$. Ans. $x = \frac{2}{a}$.

7. $4ax + a = ax + b$. Ans. $x = \frac{b - a}{3a}$.

8. $x^2 + a^2 = (x + b)^2$. Ans. $x = \frac{a^2 - b^2}{2b}$.

9. $ax + ab = a^2 + bx$. Ans. $x = a$.

10. $b(b - x) = a(a - x)$.

Solution. (1) $b^2 - bx = a^2 - ax$.
(2) $ax - bx = a^2 - b^2$.
(3) $x(a - b) = (a - b)(a + b)$.
(4) $x = ?$ Explain each step and check.

11. $x^2 - bx = (x - b)^2$. Ans. $x = b$.

12. Solve No. 11 for b.

Solution. $x^2 - bx = (x - b)^2$.
(1) $x^2 - bx = x^2 - 2bx + b^2$.
(2) $-b^2 + bx = 0$.
(3) $b^2 - bx = 0$.
(4) $b(b - x) = 0$, and $b = 0$, or $b = x$.
Explain each step.

13. $(x - a)(x - b) = x(x - a)$. Ans. $x = a$.

14. Solve No. 13 for a following the plan of No. 12.

15. $(x + a)(x - a) = x(x - b)$.

16. Solve No. 15 for b.

17. $(x + b)^2 = (x - a)^2$.

18. $(x + b)^2 = x(x + a)$.

19. $a(x + b)^2 = ax^2 + abx$.

20. Solve No. 19 for b.

21. $\frac{mx}{n} + \frac{nx}{m} = m^2 + n^2$.

22. $\frac{x}{a} + \frac{x}{2a} = 3$.

23. $\frac{x + m}{x - n} = \frac{1}{2}$.

24. $cx + a + \frac{x}{c} = \frac{x}{a}$. Ans. $x = \frac{-a^2c}{ac^2 + a - c}$.

25. $\frac{x - a}{b} - \frac{x + b}{a} = -2$.

26. $\frac{a}{x + a} + \frac{b}{x - a} = \frac{b(a - b)}{x^2 - a^2}$.

103. Literal simultaneous equations.

Exercise 89

Solve for x and y unless otherwise indicated, and check. Study the suggestions.

1. $x + y = 3a$ (1)
   $x - y = a$ (2).

Solution. Adding (1) and (2) gives $2x = 4a$.
Therefore $x = 2a$, and $y = a$. (Explain.)
Checking, $2a + a = 3a$ in (1).
$2a - a = a$ in (2).

2. $x + 3y = 7a$
   $5x - 2y = 18a$.

3. $ax + by = 1$
   $ax - by = 3$.

4. $x + y = a + b$
   $3x - 2y = a - b$.

5. $2x - y = 4a - 3b$
   $x + y = 2a + 3b$.

6. $2ax + 3by = 1$ (1)
   $3ax + 2by = 2$ (2).

Solution. $6ax + 9by = 3$. (3) Multiplying (1) by 3.
$6ax + 4by = 4$. (4) Multiplying (2) by 2.
$5by = -1$. Subtracting (4) from (3).
Ans. $y = -\frac{1}{5b}$, and $x = \frac{4}{5a}$.

7. $3x + 2y = 5a$
   $5x + 3y = 8a$.

8. $ax = by$
   $x + y = d$.

9. $ax + by = e$ (1)
   $cx + dy = f$ (2).

Solution. $acx + bcy = ce$ (3). Multiplying (1) by c.
$acx + ady = af$ (4). Multiplying (2) by a.
$bcy - ady = ce - af$ (5). Subtracting (4) from (3).
Therefore $y = \frac{ce - af}{bc - ad}$.
Solve for x by eliminating y in a similar manner.

10. $mx + ny = k$
    $ax - ky = r$.

11. $ax + by = r$
    $cx - y = n$.

104. Literal quadratic equations.

Exercise 90

Solve the following literal quadratic equations by factoring, and check:

1. $x^2 - 3ax + 2a^2 = 0.$

Solution. Factoring gives $(x - 2a)(x - a) = 0.$
Therefore $x = 2a$, and $x = a$. (Explain.)

2. $x^2 - 5mx + 6m^2 = 0.$ Ans. $x = 3m$, or $x = 2m.$

3. $x^2 + mx - 6m^2 = 0.$ Ans. $x = -3m$, or $x = 2m.$

4. $x^2 - ax - bx + ab = 0.$

Solution. (1) $(x^2 - ax) - (bx - ab) = 0.$
(2) $x(x - a) - b(x - a) = 0.$
(3) $(x - b)(x - a) = 0$, and $x = b$, or $x = a$. Ans.
Explain each step and check.

5. $3x^2 - ax - 2a^2 = 0.$ Ans. $x = a$, or $x = -\frac{2a}{3}.$

6. $2x^2 - 5ax + 2a^2 = 0.$

7. $2x^2 - 3bx + b^2 = 0.$

8. $a^2m^2 - abm - 2b^2 = 0.$ Solve for m and check.

9. $2x^2 - 5ax - 3a^2 = 0.$

10. $6a^2x^2 - 7ax - 20 = 0.$

11. $3x^2 - abx - 2a^2b^2 = 0.$

12. $2x^2 - 7ax - 15a^2 = 0.$

13. $2a^2 + am - 15m^2 = 0.$ Solve for m and check.

14. $x^2 - 2ax + a^2 = 9.$

Solution. (1) $x^2 - 2ax + a^2 - 9 = 0.$
(2) $(x - a)^2 - 9 = 0.$
(3) $(x - a - 3)(x - a + 3) = 0,$
and $x = a + 3$, or $x = a - 3$. Explain each step and check.

15. $4x^2 - 4ax + a^2 = 1.$

16. $x^2 + 6x + 9 = a^2.$

17. $(x - b)^2 = a^2.$

105. Simultaneous equations, one linear and one quadratic.

Exercise 91

Solve the following systems:

1. $x + y = 5a$ (1)
   $xy = 6a^2$ (2).

Solution. $x = 5a - y$. (3). Solving (1) for x.
$(5a - y)y = 6a^2$. (4). Substituting in (2).
$5ay - y^2 = 6a^2$, (5),
or $y^2 - 5ay + 6a^2 = 0$ (6). Ans. $y = 2a$, or $y = 3a$,
Explain each step and check. $x = 3a$, or $x = 2a$.

2. $x - y = b$
   $xy = 6b^2$.

3. $2x + y = 5m$
   $xy = 2m^2$.

4. $x + y = 2a$
   $xy = a^2 - b^2$.

5. $x + y = 2m$
   $x^2 + y^2 = 2m^2 + 2n^2$.

6. A rectangle is a inches longer than it is wide and the number of square inches in its area is $2a^2$. Find its width and length.

7. A rectangle is $2n$ inches longer than it is wide and its diagonal is $10n$ inches. Find the dimensions of the rectangle.

8. Two numbers differ by a and their product is b times the smaller. What are the numbers?

9. The second of three numbers is a more than the first and the third is b more than the first. The product of the first and second is equal to 3 times the product of the first and third. Find the numbers.

10. The numerical value of the hypotenuse of a right triangle is $a^2 + 1$ and one of the sides is $a^2 - 1$. Find the other side.

11. One of the sides of a right triangle is $2nr$ and the hypotenuse is $n^2 + r^2$. Find the other side.

Exercise 92. General Review

1. If $a = -3$ and $b = 2$, find the value of $a^3 - 2b^3$.

2. If $x = 5$ and $y = -3$, evaluate $\dfrac{x^3 + y^3}{x^2 - xy + y^2}$.

3. The dimensions of a block of wood are n, $n - 1$, and $n + 2$ inches. Find expressions for (a) the total area of the six faces, (b) the sum of the edges, and (c) the volume.

4. From the sum of $x^2 - xy + y^2$ and $2x^2 + 3xy - y^4$ subtract the sum of $4x^2 - y^2$ and $x^2 + xy - 3y^2$.

5. Expand $(x^2 - x - 2)(x^2 + x - 2)(x^4 + 5x^2 + 4)$.

6. Multiply $(a^2 - ab + b^2)(a + b)$ by $(a^2 + ab + b^2)(a - b)$.

7. Simplify $18b^2 - 2(b - 5)^2 - 3(b - 4)(b + 3)$.

8. Divide $4x^4 - 9x^2 + 30x - 25$ by $2x^2 + 3x - 5$.

9. Divide $a^2 - b^2 + 2bc - c^2$ by $a + b - c$.

10. Divide $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.

Factor the following:

11. $17a^3b^4 - 51a^5b^3 + 68a^4b^4$.
12. $36x^4y^6 - 49a^8b^2$.
13. $9a^2 - 12ab + 4b^2 - 25c^2$.
14. $a^4 - 2a^2b^2 + b^4$.
15. $3a^4 - a^2b^2 - 2b^4$.
16. $n^3 + 5n^2 - 4n - 20$.
17. $81a^4 - 16c^4$.
18. $8x^3 - 50a^2x$.

Reduce the following fractions to lower terms:

19. $\dfrac{35a^5b^4c^7}{77a^6b^3c^6}$.
20. $\dfrac{12a^3b^3 - 12a^2b^4}{16a^4b^2 - 16a^3b^3}$.
21. $\dfrac{ax^2 - ay^2}{bx^2 - 2bxy + by^2}$.
22. $\dfrac{4m^2 - 9n^2}{2m^2 + 5mn + 3n^2}$.

Supply the missing terms in the following:

23. $\dfrac{2a}{3b} = \dfrac{\quad}{12b^2c^3}$.
24. $\dfrac{3x}{x + y} = \dfrac{\quad}{x^2 - y^2}$.

25. $\dfrac{-x}{x-y} = \dfrac{\qquad}{y-x}.$ **26.** $\dfrac{3x}{1} = \dfrac{\qquad}{x^2-4y^2}.$

Simplify the following:

27. $\dfrac{2a}{2}+\dfrac{a}{3}-\dfrac{a}{4}.$ **28.** $3\frac{5}{6}+2\frac{2}{3}-3\frac{7}{12}.$

29. $\dfrac{5}{2x^2}-\dfrac{1}{6x}+\dfrac{3}{4}.$ **30.** $\dfrac{n+5}{n+6}+\dfrac{2}{3}.$

31. $\dfrac{n+1}{n^2-5n+6}+\dfrac{2n+1}{n^2-4n+3}-\dfrac{3n+1}{n^2-3n+2}.$

32. $\dfrac{1}{x}-\dfrac{1}{y}+\dfrac{1}{x+y}-\dfrac{2}{x-y}.$

33. $\dfrac{1}{2}\times\dfrac{3}{4}\times\dfrac{6}{7}\times\dfrac{8}{9}\times\dfrac{12}{15}.$ **34.** $\dfrac{2a}{3b}\times\dfrac{5bc}{6a^2}\times\dfrac{8ab}{10c^3}.$

35. $\dfrac{x^2-y^2}{x^2+5xy+6y^2}\times\dfrac{x^2-9y^2}{x^2+3xy-4y^2}.$

36. $\dfrac{a^2b^2-ab-6}{a^2b^2-ab-2}\div\dfrac{a^2b^2-9}{a^2b^2-1}.$

37. $\dfrac{ax^2-ax-6a}{bx^2-4b}\times\dfrac{x^2+x-6}{x^2-6x+9}\div\dfrac{x^2+7x+12}{x^2-8x+15}.$

Solve the following equations and check:

38. $(a-5)(a-2)-(a+4)(a-1)=34.$

39. $3(2n-1)(5n-9)-5(2n+3)(3n-4)-13=0.$

40. $(x+a-b)^2-(x-a+b)^2=(2a-2b)^2.$ Solve for x.

41. $(x+a)^3-(x-a)^3=6a(x+a)^2.$ Solve for x.

42. $\dfrac{x-1}{4}+\dfrac{x+2}{3}-\dfrac{x+5}{6}=x-8.$

43. $\dfrac{3(x+3)}{4}-\dfrac{5(x+7)}{6}+\dfrac{x+1}{3}=-2.$

44. $\dfrac{3}{n+1}+\dfrac{1}{n-1}=\dfrac{n-2}{(n-1)\,(n+1)}.$

45. $\dfrac{y+1}{2}-\dfrac{2}{y+1}=\dfrac{y}{2}+\dfrac{1}{10}.$

Solve the following equations by factoring and check all roots:

46. $(2x-1)\,(x-2)-(x+1)\,(x-3)=x+1$

47. $(3n+1)\,(n+3)-2(n-1)\,(n-2)+40=0.$

48. $(3a+2)\,(2a+3)-(a-3)\,(2a-4)=0.$

49. $\dfrac{s+1}{s^2-5s+6}-\dfrac{s-5}{s^2-7s+12}=\dfrac{2s-7}{s^2-6s+8}$

50. $\dfrac{2a+1}{3a+1}-\dfrac{a+1}{2a+1}=\dfrac{1}{12}.$

Solve the following simultaneous equations and check:

51. $\dfrac{x+y}{4}-\dfrac{x-y}{5}=3$

$\dfrac{x+8}{5}+\dfrac{y-4}{3}=2.$

52. $\frac{1}{3}(x+y)-\frac{1}{4}(x-y)=3$

$\frac{3}{4}(x-2y)+\frac{4}{5}(2x+y)=16.$

53. $\dfrac{3}{x+y}=\dfrac{2}{y+3}.$

$\dfrac{4}{x+y}+\dfrac{2}{x-y}=\dfrac{32}{x^2-y^2}.$

54. $2x+3y-z=11$
$3x-2y+2z=-1$
$4x-3y-7z=47.$

55. $2a+3b+c=3$
$5a-b+3c=-13$
$3a+2b+6c=5.$

56. $5x-2y=5$
$3x+4z=1$
$3y-2z=19.$

57. $\dfrac{4}{m}+\dfrac{3}{n}=3$

$\dfrac{2}{m}-\dfrac{6}{n}=-1.$

58. $\dfrac{1}{x}+\dfrac{1}{y}-\dfrac{1}{z}=2$

$\dfrac{1}{x}-\dfrac{1}{y}+\dfrac{1}{z}=4$

$\dfrac{1}{x}-\dfrac{1}{y}-\dfrac{1}{z}=-6.$

59. $\dfrac{3}{x}+\dfrac{5}{y}=3$

$\dfrac{6}{x}-\dfrac{1}{z}=-2$

$\dfrac{10}{y}+\dfrac{3}{z}=16.$

Solve for x and y.

60. $mx+ny=k$

$rx-sy=t.$

61. $x-y=8$
$x^2+y^2=34.$

62. $x+y=8$
$xy=-33.$

63. $x+2y=1$
$2x^2-7y^2=22.$

Solve the following formulas for each of the letters in turn:

64. $V=lwh.$ **65.** $9C=5F-160.$ **66.** $l-a=nd-d.$

67. $2s=an+ln.$ **68.** $2A=a(b_1+b_2).$

Solve the following exercises and problems by graphing:

69. James starts on a long walk into the country at the rate of 2 miles per hour. Construct the graph of the formula $d=2h$ and show by means of the graph how far from home he will be in 2 hours; 3 hours; 5 hours.

70. Henry starts at 9 A. M. on a tramp at the rate of 2 miles per hour and one hour later John follows him at the rate of 3 miles per hour. Draw the two graphs, showing the distances they walk, on the same cross section paper. Determine from the graphs when they are together. Determine from the graphs at what time each reaches a point 12 miles from home.

Suggestion. The graph showing John's distance begins on the time line at the point marked 10 A. M. The two graphs will cross, showing the time they are together and the distance they are from home.

71. The temperature on a certain day at 8 A. M. was 60°. It rose 2° per hour till 11 A. M. and then rose 3° per hour till 4 P. M. From 4 P. M. till 8 P. M. the temperature fell at the rate of 4° per hour. Draw a graph showing the temperature throughout the day.

Suggestion. Use the vertical line for the temperature line. Begin with 50°. The graph will be a broken line.

72. The temperature at 6 A. M. was 50°. It rose 3° per hour for 5 hours, then 2° per hour for 3 hours, then remained stationary for 4 hours. Draw the graph.

73. On January 1st, John has $20 and he saves $5 per month throughout the year. James begins the year with $5 and saves $10 per month throughout the year. Draw the graphs on the same cross section paper and determine when they have the same amount.

74. Two towns, *A* and *B*, are 100 miles apart. An automobile starts from *A* toward *B* at the rate of 15 miles per hour and at the same time a motor truck starts from *B* toward *A* at the rate of 10 miles per hour. Draw the graphs and determine how far the machines are from *A* when they meet.

Suggestion. The graph representing the distance of the motor truck begins at the point on the distance line marked 100 miles and slopes downward.

75. A man walks a certain distance and returns. On the outward trip he makes 3 miles per hour and on the homeward trip 2 miles per hour. Altogether he walks 10 hours. Find the number of miles.

Solve the following simultaneous equations by graphing:

76. $x + y = 8$, $x - y = 2.$

77. $3x + 2y = 12$, $2x + 3y = 13.$

78. $2x + 3y = -6$, $5x - 2y = 4.$

79. $x + 2y = 9$, $2x - y = 8.$

80. $3x - 5y = 9$, $4x + y = -11.$

Solve the following:

81. The second of two numbers is 3 more than 2 times the first, and their sum is 15. Find the numbers.

82. Find four consecutive numbers whose sum is 50.

83. Find four consecutive even numbers whose sum is 60.

84. Find three consecutive numbers such that if the first be subtracted from the sum of the second and third, the remainder is 23.

85. The sum of three numbers is 59. The second is 3 more than twice the first, and the third is 2 more than 3 times the first. Find the numbers.

86. The second of three numbers is 5 more than twice the first, and the third is 1 more than 4 times the first. If the first is subtracted from the sum of the second and third, the remainder will be 41. Find the numbers.

87. If the third of three consecutive numbers be subtracted from the sum of the other two, the remainder is 29. Find the numbers.

88. The second of three numbers is 1 less than 2 times the first, and the third is 10 more than 2 times the first. Find the numbers, if the third is equal to the sum of the first and second.

89. It is 206 miles farther from New York to Buffalo than from New York to Boston, and from New York to Chicago is 32 miles more than twice the distance from New York to Buffalo. Find these three distances if their sum is 1586 miles.

90. The Nile River is 4 times the length of the Rhine, and the length of the Amazon is 100 less than that of the Nile. Find the length of each river if the sum of their lengths is 7550.

91. If a certain number is increased by one-half of itself and one-third itself, the sum is 33. What is the number?

92. Find that number which, when divided by 4, is equal to one-fifth of the sum of itself and 8.

93. Find three consecutive numbers such that the first plus one-half the second minus one-third the third will equal 15.

94. The product of two consecutive numbers is 33 more than the square of the smaller. What are the numbers?

95. Find three consecutive numbers such that the square of the first is 41 less than the product of the other two.

96. A rectangle is 3 inches longer than wide. If the length is increased 3 inches and the width is decreased 2 inches, the area will be decreased 4 square inches. Find the dimensions.

97. A picture is 3 inches longer than it is wide. The width of the frame is 2 inches and the area of the picture and frame is 100 square inches more than the area of the picture alone. Find the dimensions of the picture.

98. Find three consecutive numbers such that the product of the first and second is 2 more than 7 times the third.

99. Find three consecutive numbers such that the sum of the squares of the first two is 5 more than the square of the third.

100. A number is composed of two digits. The unit's digit is 4 less than the ten's digit. If the number is divided by the sum of the digits, the quotient is 7. Find the number.

CHAPTER VIII

POWERS AND ROOTS

106. Powers of monomials. Since $a^m \cdot a^n = a^{m+n}$ (§ 40), then $a^m \cdot a^m = a^{2m}$. But $a^m \cdot a^m$ is the square of a^m. Or, $a^m \cdot a^m = (a^m)^2 = a^{2m}$.

∴ **Rule.** *The square of a literal number is found by doubling its exponent.*

Similarly, $(a^m)^3 = a^{3m}$. In general $(a^m)^n \equiv a^{mn}$.

Rule. *To raise a literal number to a given power, multiply the exponent of the number by the index of the power.*

$(a^n \cdot b^n)^2 = (a^n)^2 \cdot (b^n)^2 = a^{2n}b^{2n}$, and $(2a^mb^n)^3 = 2^3a^{3m}b^{3n}$.

Rules. I. *To raise a monomial expression to a given power, raise each factor of the monomial to the required power.*

II. *To raise a fraction to a given power, raise each term of the fraction to the required power.*

Note. In this course in Algebra only positive integral exponents are considered.

Illustrative exercises.

1. $(2a^3b^2c)^2 = 4a^6b^4c^2$. **2.** $(3a^2b)^m = 3^ma^{2m}b^m$.

3. $(-3a^nb^2)^3 = -27a^{3n}b^6$. **4.** $\left(\dfrac{4a^mb}{c}\right)^n = \dfrac{4^na^{mn}b^n}{c^n}$.

Exercise 93

Simplify the following by performing the indicated operations:

1. $(3a^2b^3)^2$.
2. $(4x^2y^2z^2)^2$.
3. $(5xy^3z^4)^2$.
4. $\left(\dfrac{2a^2b}{5}\right)^3$.
5. $(3m^2n^2x)^3$.
6. $\left(\dfrac{2a^3b^3c}{3x}\right)^4$.
7. $(-2x^2y^3z^4)^3$.
8. $(-3x^2yz^2)^3$.
9. $(-4x^4y^2z)^2$.
10. $\left(\dfrac{-2m^3n^5y}{3}\right)^3$.
11. $\left(\dfrac{-3a^2b^3c}{2}\right)^4$.
12. $(10r^2s^5)^3$.
13. $\left(\dfrac{5ab}{2}\right)^m$.
14. $\left(\dfrac{2x^2y}{3}\right)^2$.
15. $(3^2a^3b^4)^n$.
16. $\left(\dfrac{2^ma^mb}{c}\right)^2$.
17. $(3x^my^n)^2$.
18. $(-2a^nb^m)^3$.
19. $(-5x^2y^m)^2$.
20. $(x^my^m)^n$.

107. Powers of binomials. Quite often it has been possible, as in the last section, to solve a general problem or exercise and call the result the **type form.** This type form, or formula, may be used to get the results of all other exercises that are similar in form and operation.

All **type forms** are **identities** (see § 18) and therefore true for all possible values of the literal numbers.

The student is acquainted with the type form $(a+b)^2 \equiv a^2+2ab+b^2$.

Since a and b may have any values, such as $2x$ and $3y$, this formula may be used to expand $(2x+3y)^2$ as follows: $(2x+3y)^2=(2x)^2+2(2x)\ (3y)+(3y)^2=4x^2+12xy+9y^2$. Check the result by actual multiplication.

Any power of any binomial may be found by raising the general binomial $(a+b)$ to the required power, and then substituting the terms of the given binomial for a and b.

The following are the powers of $(a+b)$ to $(a+b)^5$. Verify each type form by actual multiplication.

$(a+b)^2 \equiv a^2+2ab+b^2.$
$(a+b)^3 \equiv a^3+3a^2b+3ab^2+b^3.$
$(a+b)^4 \equiv a^4+4a^3b+6a^2b^2+4ab^3+b^4.$
$(a+b)^5 \equiv a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$

And if $(a-b)$ is raised to the same powers, we get:

$(a-b)^2 \equiv a^2-2ab+b^2.$
$(a-b)^3 \equiv a^3-3a^2b+3ab^2-b^3.$
$(a-b)^4 \equiv a^4-4a^3b+6a^2b^2-4ab^3+b^4.$
$(a-b)^5 \equiv a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5.$

Compare the type forms of $(a+b)$ with those of $(a-b)$ and state the evident law for the signs of the terms.

How is the number of terms of the expansion related to the exponent of $(a+b)$?

How is the numerical coefficient of the second term of the expanded form related to the exponent of $(a+b)$?

Can you determine succeeding numerical coefficients?

Write the expanded product of $(2x+y)^3$ by substituting in the proper type form and then removing the parentheses.

Ans. $8x^3+12x^2y+6xy^2+y^3.$

Exercise 94

Expand by substituting in the type forms:

1. $(x+3y)^2.$	**7.** $(2m-3y)^3.$	**13.** $\left(\frac{x}{2}-\frac{y}{4}\right)^2.$
2. $(2d-5e)^2.$	**8.** $(2x+y)^4.$	**14.** $(2a^2-3x^3)^2.$
3. $(2m+n)^3.$	**9.** $(3x-5y)^3.$	**15.** $\left(\frac{a^2}{b^2}-\frac{x^2}{y^2}\right)^2.$
4. $(x+y)^5.$	**10.** $\left(x-\frac{y}{2}\right)^2.$	**16.** $\left(\frac{a}{x}+\frac{x}{a}\right)^2.$
5. $(x-1)^5.$	**11.** $\left(2x+\frac{y}{3}\right)^3.$	**17.** $\left(\frac{a}{x}+\frac{x}{a}\right)^3.$
6. $(a-2b)^3.$	**12.** $\left(3a-\frac{2y}{3}\right)^3.$	**18.** $\left(\frac{a}{x}-\frac{x}{a}\right)^4.$

108. Roots of numbers. The **square root** of a number is one of the two equal factors of the number. Similarly, the **cube root** of a number is one of the three equal factors of the number. In general, the ***n*th root** of a number is one of the n equal factors of the number.

The character $\sqrt{}$ (r from the word radical, or root) and the vinculum — over the number is used to indicate a root. A small figure above the symbol indicates what root is to be taken. For square root the figure is omitted.

$\sqrt{25}$, $\sqrt[3]{25}$, $\sqrt[3]{(a+b)}$ are to be read, respectively, "the square root of 25," "the cube root of 25," and "the cube root of the expression $a+b$."

Since the square of a^m is a^{2m}, the square root of a^{2m} is a^m, or $\sqrt{a^{2m}}=a^m$.

Rule. *The square root of a literal number is found by dividing the exponent by* 2.

Similarly, the cube root of a literal number is found by dividing the exponent by 3. That is, $\sqrt[3]{a^6}=a^2$, etc.

Note. Since $(a)(a)=a^2$ and $(-a)(-a)=a^2$, the square root of a^2 may be either $+a$ or $-a$. This is expressed by the symbol $\pm$, to be read "plus or minus." Unless otherwise specified, we shall use only $+a$ as $\sqrt{a^2}$.

Rules. I. *To find the indicated root of a monomial expression, find the required root of the numerical coefficient and divide the exponent of each of the literal numbers by the index of the root.*

II. *To find the indicated root of a fraction, find the required root of both numerator and denominator.*

Illustrative examples.

1. $\sqrt{9a^2b^4}=3ab^2$.
2. $\sqrt[3]{27x^3y^6}=3xy^2$.
3. $\sqrt[3]{-8a^6x^9}=-2a^2x^3$.
4. $\sqrt{\dfrac{a^2}{b^2}}=\dfrac{a}{b}$.
5. $\sqrt{9\cdot 25\cdot 36}=3\cdot 5\cdot 6=90$.

Exercise 95

Find the indicated root of the following:

1. $\sqrt{a^4}$. 2. $\sqrt[3]{a^9}$. 3. $\sqrt[3]{a^{12}}$. 4. $\sqrt[3]{a^{15}}$. 5. $\sqrt{64}$.

6. $\sqrt[3]{8}$. 7. $\sqrt{16}$. 8. $\sqrt{81}$. 9. $\sqrt[3]{125}$. 10. $\sqrt[3]{27}$.

11. $\sqrt{9a^2b^2c^4}$. 12. $\sqrt[3]{27x^3y^6z^9}$. 13. $\sqrt{81x^4y^8z^{12}}$.

14. $\sqrt[3]{-8x^3y^9}$. 15. $\sqrt[3]{-27x^6y^9}$. 16. $\sqrt{x^{12}y^{18}}$.

17. $\sqrt{16 \cdot 25 \cdot 49}$. 18. $\sqrt[3]{8 \cdot 27 \cdot 125}$. 19. $\sqrt[3]{-125y^3}$.

20. $\sqrt[3]{(-8)(-125)}$. 21. $\sqrt{484} = \sqrt{4 \cdot 121} = 2 \cdot 11 = 22$.

22. $\sqrt{576}$. 23. $\sqrt{441}$. 24. $\sqrt{1089}$.

25. $\sqrt{1296}$. 27. $\sqrt{\dfrac{25x^2}{36y^2}}$. 28. $\sqrt{\dfrac{9a^4x^6}{16b^2y^8}}$.

26. $\sqrt{3025}$.

Note. We cannot find an even root of a negative number. Why?

109. Square root of polynomials. Since $(a+b)^2 \equiv a^2 + 2ab + b^2$, then $\sqrt{a^2+2ab+b^2} \equiv a + b$.

A study of this type form helps to make clear the process of finding the square root of a polynomial which is a perfect square. Arrange the work as in the following **illustrative examples:**

1.
$$\begin{array}{lr|l}
 & a^2 + 2ab + b^2 & \underline{(a+b} \\
 & a^2 & \\ \hline
\text{Trial divisor} = 2a & +2ab + b^2 & \\
\text{Complete divisor} = 2a + b, & +2ab + b^2 & \\ \hline
\end{array}$$

Observations. (1) The polynomial is arranged, as far as possible, with reference to one letter. (a in the type form.)

(2) The square root of a^2, or a, is obtained by observation and is placed at the right as the first term of the root.

(3) Subtracting a^2 gives a remainder of $2ab + b^2$. The second term of the root may be obtained by dividing $2ab$ by $2a$, since in the type form $2ab$ is twice the produce of the terms of $(a+b)$.

(4) Taking $2a$ as the trial divisor and adding b to it, we have the complete divisor $2a + b$.

(5) The complete divisor is multiplied by b. Since there is no remainder, therefore $\sqrt{a^2 + 2ab + b^2} \equiv a + b$.

2. Find one square root of: $a^4 + 11a^2 - 6a - 6a^3 + 1$.
Arranging in descending powers of a,

$$\begin{array}{ll}
 & \underline{\qquad\qquad a^2 - 3a + 1} \\
 & a^4 - 6a^3 + 11a^2 - 6a + 1 \\
 & \underline{a^4} \\
\text{Trial divisor} = 2a^2. & \quad -6a^3 + 11a^2 \\
\text{Complete divisor} = 2a^2 - 3a. & \quad \underline{-6a^3 + \ 9a^2} \\
\text{Trial divisor} = 2(a^2 - 3a) = 2a^2 - 6a. & \qquad\qquad 2a^2 - 6a + 1 \\
\text{Complete divisor} = 2a^2 - 6a + 1. & \qquad\qquad \underline{2a^2 - 6a + 1}
\end{array}$$

Exercise 96

Find the square root of each of the following:

1. $x^2 + 2xy + y^2$. **2.** $n^2 + 8n + 16$.

3. $4a^2 - 12ab + 9b^2$.

4. $x^4 + 2x^3 + 3x^2 + 2x + 1$.

5. $a^4 - 4a^3 + 6a^2 - 4a + 1$.

6. $n^4 + 8n^3 + 12n^2 - 16n + 4$.

7. $a^2 - 2ab + b^2 - 2ac + 2bc + c^2$.

8. $9x^2 - 24xy + 16y^2 + 30xz - 40yz + 25z^2$.

9. $16a^4 + 8a^3b - 23a^2b^2 - 6ab^3 + 9b^4$.

10. $a^8 - 2a^7 + 3a^6 - 4a^5 + 5a^4 - 4a^3 + 3a^2 - 2a + 1$.

11. $4x^6 - 12x^5y + 5x^4y^2 + 10x^3y^3 - 5x^2y^4 - 2xy^5 + y^6$.

12. $n^6 - 6n^5 + 15n^4 - 20n^3 + 15n^2 - 6n + 1$.

13. $a^6 - 12a^5 + 60a^4 - 160a^3 + 240a^2 - 192a + 64$.

14. $5n^2 + 4n^4 + 2n + 1 + 4n^3$.

15. $a^4 + b^4 - 2ab^3 - 2a^3b + 3a^2b^2$.

110. Square root of arithmetical numbers. An arithmetical number of two or more places, or digits, may be considered as a binomial.

$45 = 40 + 5$, and $(45)^2 \equiv (40 + 5)^2 \equiv (40)^2 + 2(40)(5) + 5^2 \equiv 1600 + 400 + 25 \equiv 2025$.

Note that the square of any number of one digit is a number of not more than two digits, and that the square of any number of two digits is a number of more than two and not more than four digits.

Verify the above statement by finding the squares of 1, 9, 10, and 99. Also square 100 and 999.

How many digits has the square of a number of three places?

The square root of a perfect square of three or four digits will be an integer of two digits, and the square root of a perfect square of five or six places will be an integer of three digits.

Verify similar conclusions for decimal numbers by finding the squares of, .1, .9, .01, and .99.

The number of digits in the square root of an arithmetical number is indicated by grouping into periods of two figures each, to the right and left from the decimal point.

As, for example, $\sqrt{47961}$ is an integer of three places.

Illustrative examples. 1. Find the square root of 4225.

$$\begin{array}{ll} & 4225 \mid \underline{60+5=65.} \\ & \underline{3600} \\ \text{Trial divisor} = 2(60) = 120 & \quad 625 \\ \text{Complete divisor} = 120 + 5 = 125 & \quad \underline{625 = (120+5)\ 5.} \end{array}$$

Observations. (1) The root will be an integer of 2 places (2) 4225 is between 3600 and 4900, therefore the square root will be between 60 and 70, or the root will be $60 + n$, with n representing the second digit of the root.

(3) Subtracting $(60)^2$ leaves 625.

(4) As with literal polynomials, the second term of the root is determined by dividing the remainder by the trial divisor, which is 2(60), giving as a quotient 5.

(5) The complete divisor is $2(60) + 5$, or 125. As there is no remainder, $\sqrt{4225} = 65$.

2. Find the square root of 54756.

If the unnecessary zeros are omitted, the work may be arranged as follows:

$$\begin{array}{ll} & 5'47'56 \mid \underline{234} \\ & 4 \\ \text{Trial divisor} = 2 \times 20 = 40. & \quad 147 \\ \text{Complete divisor} = 40 + 3 = 43. & \quad \underline{129} \\ \text{Trial divisor} = 2 \times 230 = 460. & \quad\ 1856 \\ \text{Complete divisor} = 460 + 4 = 464. & \quad\ \underline{1856} \end{array}$$

Note. The square root of 547.56 is 23.4 and the square root of 5.4756 is 2.34.

Exercise 97

Find one square root of each of the following:

1. 169. **2.** 225. **3.** 529. **4.** 1681. **5.** 2809.

6. 7225. **7.** 64516. **8.** 156816. **9.** 104329. **10.** 12823561.

11. 20.25. **12.** 156.25. **13.** 6512.49. **14.** 5227.29.

15. 165649. Ans. 407.

16. 644809. Ans. 803.

17. 7856809. Ans. 2803.

18. 16216729. Ans. 4027.

19. 800.89. Ans. 28.3.

20. 1823.29. Ans. 42.7.

21. 150.0625. Ans. 12.25.

22. 1372.7025. Ans. 37.05.

23. .390625. Ans. .625.

24. .104329. Ans. .323.

25. .002401. Ans. .049.

26. .00015129. Ans. .0123.

Find the square root of each of the following correct to the nearest hundredth:

27. 347.

Solution:

	3′47.′00′00′00	18.627
	1	
$2 \times 10 = 20$	247	
$20 + 8 = 28$	224	
$2 \times 180 = 360$	2300	
$360 + 6 = 366$	2196	
$2 \times 1860 = 3720$	10400	
$3720 + 2 = 3722$	7444	
$2 \times 18620 = 37240$	295600	
$37240 + 7 = 37247$	260729	
	34871	

The answer is 18.63. Explain why it was necessary to find the third decimal place in order to get the result correct to the nearest hundredth.

28. 7. **29.** 70. **30.** 5. **31.** 50.

32. 3.1416. **33.** 17. **34.** 22. **35.** 77.

RADICALS

111. Definitions. The use of the symbol, $\sqrt{\ }$, called the **radical sign,** to indicate that a root is wanted was explained in § 108.

Any expression which contains an indicated root is called a **radical expression.** $3\sqrt{2}$, $4-2\sqrt{3}$, $\sqrt{2}+\sqrt{3-x}$, etc., are radical expressions.

The expression under the radical sign is called the **radicand.**

Integers and fractions whose terms are integers (that is, do not contain radicals) are called **rational numbers.** Any number which can be reduced to an integer, or to a fraction whose terms are integers, is a rational number.

$\sqrt{4}$ and $\sqrt[3]{27}$ are rational numbers.

Also 5, 23, $\frac{1}{2}$, $\frac{2}{3}$, etc., are rational numbers.

An indicated root which cannot be reduced to a rational number is called a **surd,** or an **irrational number.**

$\sqrt{5}$ and $\sqrt[3]{17}$ are surds.

The product of a rational number and a surd, such as $2\sqrt{3}$, is called a **mixed surd.**

The **order,** or **degree,** of the surd is determined by the index of the root. $\sqrt[3]{7}$ is a surd of the **third degree.** $\sqrt{5}$ is a surd of the **second degree.**

112. Simplification of surds. It is frequently possible to change, or simplify, the form of the surd without changing its value.

Since $(a \cdot b)^3 = a^3b^3$, then $\sqrt[3]{a^3b^3} = \sqrt[3]{a^3} \cdot \sqrt[3]{b^3} = ab$.

Similarly, $\sqrt{24} = \sqrt{4 \cdot 6} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$.

Also, $\sqrt{36} = 6$. Or $\sqrt{36} = \sqrt{4}\sqrt{9} = 2 \cdot 3 = 6$.

Rules. I. *The root of a product may be obtained by finding the separate roots of the factors of the product and multiplying the results.*

II. *An indicated root may be simplified by factoring the number under the radical into two factors, one of which is a perfect power of the same degree as the root, and then finding the root of the power.*

Illustrative examples.

1. $\sqrt{8} = \sqrt{4}\,\sqrt{2} = 2\sqrt{2}.$
2. $\sqrt{12} = 2\sqrt{3}.$ Explain.
3. $2\sqrt{x^3} = 2\sqrt{x^2}\cdot\sqrt{x} = 2x\sqrt{x}.$
4. $\sqrt{50x^3y} = \sqrt{25x^2}\cdot\sqrt{2xy} = 5x\sqrt{2xy}.$
5. $3\sqrt[3]{40} = 3\sqrt[3]{8}\cdot\sqrt[3]{5} = 6\sqrt[3]{5}.$

Exercise 98

Simplify the following so that no factor of the number under the radical sign is a power of the same degree as the root:

1. $\sqrt{18}.$ **2.** $\sqrt{28}.$ **3.** $\sqrt{32}.$ **4.** $5\sqrt{45}.$ **5.** $\sqrt[3]{40}.$

6. $\sqrt[3]{24}.$ **7.** $2\sqrt{50a^2}.$ **8.** $3\sqrt{48x^3a^4}.$

9. $\sqrt{27a^4b^5x^2}.$ **10.** $\sqrt[3]{27a^4}.$ **11.** $5\sqrt[3]{x^5}.$

12. $\sqrt[3]{a^3b^4}.$ **13.** $\sqrt[3]{32a^5x^6}.$ **14.** $\sqrt[3]{a^5b^6c^7}.$

15. $\sqrt{32x}.$ **16.** $2\sqrt{64x^6}.$ **17.** $\sqrt{96a^6b^7}.$

18. $\sqrt{(a-b)^2x}.$ Ans. $(a-b)\sqrt{x}.$

19. $\sqrt{x^3 - 2x^2y + xy^2}.$ **20.** $\sqrt{(a^2-b^2)(a-b)}.$

21. $\sqrt{9(x^2-2xy+y^2)(a-b)}.$ **22.** $\sqrt[3]{(a-b)^4}.$

23. $\sqrt{2a^2 - 4ab + 2b^2}.$

24. $\sqrt{(x-y)(2x^2-3xy+y^2)}.$

25. $\sqrt{32(x-y)(x^2-y^2)}.$

26. $\sqrt{(3x^2-5xy+2y^2)(4x^2-7xy+3y^2)}.$

113. Simplification of fractional surds. A fractional surd is simplified when the number remaining under the radical is not in fractional form and contains no factor that is a power of the same degree as the indicated root.

In order to simplify a fractional surd, it is necessary first to change the fraction under the radical to an equivalent fraction whose denominator is a perfect power of the same degree as the root. This is done by multiplying the numerator and denominator of the fraction by the same number.

Illustrative examples.

1. $\sqrt{\frac{1}{2}}=\sqrt{\frac{2}{4}}=\sqrt{\frac{1}{4}\cdot 2}=\frac{1}{2}\sqrt{2}.$
2. $\sqrt{\frac{8}{3}}=\sqrt{\frac{24}{9}}=\sqrt{\frac{1}{9}}\cdot\sqrt{24}=\sqrt{\frac{1}{9}}\cdot\sqrt{4}\cdot\sqrt{6}=\frac{2}{3}\sqrt{6}.$
3. $\sqrt{\frac{1}{a-b}}=\sqrt{\frac{a-b}{(a-b)^2}}=\sqrt{\frac{1}{(a-b)^2}}\cdot\sqrt{a-b}=\frac{\sqrt{a-b}}{a-b}.$

Exercise 99

Simplify the following:

1. $\sqrt{\frac{1}{3}}.$ 2. $\sqrt{\frac{2}{5}}.$ 3. $\sqrt{\frac{4}{5}}.$ Ans. $\frac{2}{5}\sqrt{5}.$ 4. $\sqrt{\frac{12}{5}}.$

5. $\sqrt{\frac{2a}{3b}}.$ 6. $\frac{2}{3}\sqrt{\frac{1}{2}}.$ 7. $\frac{2}{3}\sqrt{\frac{18}{5}}.$ 8. $\sqrt{\frac{9}{8x^2}}.$

9. $5\sqrt{\frac{1}{a}}.$ 10. $\sqrt{\frac{x}{y}}.$ 11. $\sqrt{\frac{ab}{xy}}.$ 12. $\sqrt{\frac{3}{n}}.$

13. $\sqrt{\frac{5}{6}}.$ 14. $\sqrt{\frac{5}{7}}.$ 15. $\frac{1}{3}\sqrt{\frac{3}{4}}.$

114. Mixed surds to entire surds. A mixed surd may be changed into an entire surd by reversing the operations of the preceding section.

Illustrative examples.

1. $3\sqrt{5}=\sqrt{9}\cdot\sqrt{5}=\sqrt{45}.$ 2. $2\sqrt[3]{2}=\sqrt[3]{8}\cdot\sqrt[3]{2}=\sqrt[3]{16}.$
3. $(a-b)\sqrt{n}=\sqrt{(a-b)^2}\cdot\sqrt{n}=\sqrt{(a-b)^2n}.$
4. $\frac{1}{2}\sqrt{2}=\sqrt{\frac{1}{4}}\cdot\sqrt{2}=\sqrt{\frac{2}{4}}=\sqrt{\frac{1}{2}}.$

Exercise 100

Change the following to entire surds:

1. $3\sqrt{2}$. **2.** $3\sqrt{3}$. **3.** $3\sqrt{10}$. **4.** $2\sqrt{5}$. **5.** $2\sqrt[3]{3}$.

6. $3\sqrt[3]{2}$. **7.** $2\sqrt{b}$. **8.** $2a\sqrt{a}$. **9.** $x\sqrt[3]{y}$. **10.** $ab\sqrt{xy}$.

11. $xy\sqrt[3]{x}$. **12.** $a\sqrt[3]{b}$. **13.** $3\sqrt{ab}$. **14.** $2xy\sqrt{xy}$.

15. $(x-y)\sqrt{2}$. **16.** $(a-b)\sqrt{a+b}$.

17. $3x^2y\sqrt{x-y}$. **18.** $(m-n)\sqrt[3]{3}$. **19.** $\frac{2}{3}\sqrt{3}$.

20. $\frac{1}{a}\sqrt{a}$. **21.** $\frac{3}{4}\sqrt{7}$. **22.** $\frac{a}{b}\sqrt{ab}$. **23.** $\frac{1}{a-b}\sqrt{a-b}$.

115. Reduction of radicals to their approximate decimal equivalents. The approximate square root of a number that is not a perfect square may be found by simplifying the radical expression and then substituting the decimal equivalent for the remaining radical.

For the exercises of this section, use $\sqrt{2}=1.414$, $\sqrt{3}=1.732$, $\sqrt{5}=2.236$, and $\sqrt{6}=2.449$.

Illustrative examples.

1. $\sqrt{12}=\sqrt{4}\cdot\sqrt{3}=2(1.732)=3.464.$
2. $\sqrt{50}=\sqrt{25}\cdot\sqrt{2}=5(1.414)=7.07.$
3. $\sqrt{\frac{1}{2}}=\sqrt{\frac{2}{4}}=\sqrt{\frac{1}{4}}\cdot\sqrt{2}=\frac{1}{2}(1.414)=.707.$
4. $\sqrt{\frac{2}{3}}=\sqrt{\frac{6}{9}}=\sqrt{\frac{1}{9}}\cdot\sqrt{6}=\frac{1}{3}(2.449)=.816.$

Exercise 101

Reduce the following to their decimal equivalents:

1. $\sqrt{8}$. **2.** $\sqrt{75}$. **3.** $\sqrt{32}$. **4.** $\sqrt{54}$. **5.** $\sqrt{200}$.

6. $\sqrt{\frac{1}{3}}$. **7.** $\sqrt{\frac{4}{5}}$. **8.** $\frac{2}{3}\sqrt{18}$. **9.** $\sqrt{\frac{3}{2}}$. **10.** $\frac{1}{2}\sqrt{\frac{2}{3}}$.

11. $\sqrt{80}$. **12.** $\sqrt{96}$. **13.** $\frac{1}{3}\sqrt{72}$. **14.** $25\sqrt{\frac{1}{5}}$. **15.** $\sqrt{98}$.

116. Equations which reduce to the form $x^2 = n$. We will be able to solve equations of the form $x^2 = n$, if we assume the following:

Axiom VIII. *Like powers or like roots of equals are equal.*

Illustrative examples.

1. $x^2 = 36$. $x = \pm 6$. Why are there two answers?
2. $x^2 = 40$. $x = \pm \sqrt{40}$. $x = \pm 6.324$.

Exercise 102

Solve the following for x:

1. $x^2 = 25$.
2. $x^2 = 3025$.
3. $x^2 = 6889$.
4. $x^2 = 5$.
5. $x^2 = 20$.
6. $x^2 = 75$.
7. $x^2 = 9a^2$.
8. $x^2 = (a + b)^2$.
9. $x^2 = 8a^2$.
10. $(x + 3)^2 + (x - 3)^2 = 68$.
11. $(x + 5)^2 + (x - 5)^2 = 178$.
12. $(x + a)^2 + (x - a)^2 = 20a^2$.
13. $3(x + 2)(x + 1) = (x + 5)(x + 4) - 6$.
14. $3(x - 5)(x + 4) - (x - 6)(x + 3) = 30$.
15. $\dfrac{x-1}{x+1} + \dfrac{x+1}{x-1} = \dfrac{5}{2}$.
16. $\dfrac{x-3}{x+3} + \dfrac{x+3}{x-3} = \dfrac{10}{3}$.
17. $\dfrac{x^2+3}{4} - \dfrac{x^2-1}{2} = x^2 - 10$.
18. $\dfrac{30}{8-x} = 4 + \dfrac{14}{3x-2}$.
19. $\dfrac{x+2}{x-2} - \dfrac{2x+3}{x+2} = \dfrac{5x-8}{x^2-4}$.
20. $\dfrac{x+1}{x^2-5x+6} - \dfrac{x+2}{x^2-4x+3} = \dfrac{x+3}{x^2-3x+2}$.

Exercise 103. Problems

1. Find the length of the diagonal of a square whose sides are each 5 inches.

Suggestion. Draw a square and notice that the diagonal and two sides form a right triangle. Then if d represents the number of inches in the diagonal, $d^2 = 5^2 + 5^2$. Ans. 7.07 + inches.

2. A regulation baseball diamond is a square 90 feet on each side. How far is it from second base to home plate? How far from first base to third base?

3. A park is in the shape of a rectangle, 200 feet by 300 feet. How much shorter is a path that runs diagonally across the park than the walk around the two sides of the park?

4. An acre of land is an area of 160 square rods. Each square rod is equivalent in area to a square one side of which is 1 rod or 16.5 feet. How many square feet in a square rod? How many square feet in an acre?

5. A lot in the shape of a perfect square has an area of one acre. How many rods in the length of one side of the lot? How many feet?

6. A lot is twice as long as it is wide and has an area of one acre. How many feet in the length of each side?

Suggestion. $(x)(2x) = 43560$.

7. How long a straight line can be drawn on a sheet of paper 9 inches by 14 inches?

8. What is the length of one side of a square whose diagonal is 100 feet?

Suggestion. $x^2 + x^2 = 100^2$.

9. A pond is in the shape of a circle and covers an acre of land. What is the length of its diameter in feet?

Suggestion. $\pi r^2 = 43560$. Using $\pi = \frac{22}{7}$, show that $r^2 = 13860$.

10. What is the circumference of the pond of No. 9? (Use $c = 2\pi r$).

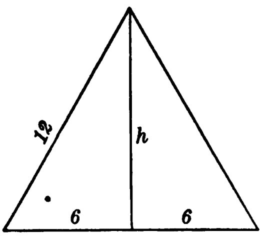
11. Find the diameter of a circle in inches whose area is one square foot.

12. A square and a circle have the same area, 100 square inches. Find the difference between their perimeters correct to .01.

13. The sides of an equilateral triangle are each 12 inches. What is the altitude of the triangle?

Note. It is proved in Geometry that a line drawn from the vertex of an equilateral triangle to the middle point of the base is perpendicular to the base. Therefore this line is the altitude of the whole triangle and divides it into two right triangles.

Show that $h^2 + 6^2 = 12^2$.



14. Find the area of the triangle of No. 13.

15. Find the altitude and the area of an equilateral triangle each of whose sides is 8 inches.

16. If the altitude of an equilateral triangle is 10 inches, find the length of a side.

Suggestion. Let x represent the number of inches in half the length of a side. Show that $10^2 + x^2 = (2x)^2$.

17. One side of a right triangle is twice the other and the hypotenuse is 12 feet. Find the lengths of the two sides.

18. The time for the fall of a bomb, when the height from which it is dropped is known, is given by the formula $t = \frac{\sqrt{H}}{4} + \frac{H}{9000}$ where t is the time in seconds and H the height in feet.

If a plane is flying at a height of 9000 feet, what will be the time of fall of a bomb correct to .1 of a second?

19. If a bomb is dropped from the height of 20,000 feet, in what time will it reach the ground?

117. Addition and subtraction of radicals.

Surds of the same degree and the same radicand are **similar surds.**

$\sqrt{a}$, $2\sqrt{a}$, and $\frac{1}{2}\sqrt{a}$ are similar surds.

Likewise, $\sqrt[3]{2}$, $5\sqrt[3]{2}$, and $(a+b)\sqrt[3]{2}$. Similar surds and surds that can be changed to similar surds may be combined into a single surd.

Illustrative examples.

1. $\sqrt{2}+5\sqrt{2}-3\sqrt{2}=3\sqrt{2}.$
2. $\sqrt{12}+\sqrt{3}-\sqrt{75}=2\sqrt{3}+\sqrt{3}-5\sqrt{3}=-2\sqrt{3}.$
3. $\sqrt{5}+\sqrt{20}-2\sqrt{\frac{1}{5}}=\sqrt{5}+2\sqrt{5}-\frac{2}{5}\sqrt{5}=\frac{13}{5}\sqrt{5}.$
4. $\sqrt{a}+\sqrt{ab^2}+\sqrt{a^2b^2}=\sqrt{a}+b\sqrt{a}+ab\sqrt{a}=(1+b+ab)\sqrt{a}.$

Exercise 104

Simplify and combine like terms:

1. $\sqrt{18}+\sqrt{32}-\sqrt{8}.$
2. $\sqrt{12}+\sqrt{27}-2\sqrt{3}.$
3. $\sqrt{32}-\sqrt{50}+\sqrt{72}.$
4. $\sqrt{75}-\sqrt{27}+6\sqrt{\frac{1}{3}}.$
5. $2\sqrt{\frac{1}{2}}+\sqrt{8}-\sqrt{72}.$
6. $9\sqrt{\frac{1}{3}}+\sqrt{12}-2\sqrt{75}.$
7. $2\sqrt{a}+\sqrt{a^3}+\sqrt{a^5}.$
8. $\sqrt{x^3}+\sqrt{xy^2}+\sqrt{9x}.$
9. $2\sqrt{25b}-\sqrt{4b}+\sqrt{9b}.$
10. $6\sqrt{\frac{2}{3}}+\sqrt{24}-\sqrt{54}.$
11. $\sqrt[3]{2}+\sqrt[3]{54}-\sqrt[3]{16}.$
12. $\sqrt[3]{a^2}+\sqrt[3]{a^5}-\sqrt[3]{a^2b^3}.$
13. $3\sqrt{a+b}-\sqrt{4a+4b}+\sqrt{a^3+a^2b}.$
14. $\frac{2}{3}\sqrt{\frac{3}{2}}+\frac{3}{2}\sqrt{\frac{2}{3}}.$

118. Multiplication of monomial surds. From the preceding sections it will be observed that surds of the same degree may be multiplied and the product written as a single surd or as a rational expression.

Illustrative examples.

1. $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}.$
2. $\sqrt{3}\cdot\sqrt{6}=\sqrt{18}=3\sqrt{2}.$
3. $2\sqrt{5}\cdot 3\sqrt{20}=6\sqrt{100}=60.$
4. $\frac{1}{3}\sqrt{2}\cdot\frac{1}{2}\sqrt{3}\cdot\frac{1}{4}\sqrt{\frac{1}{2}}=\frac{1}{24}\sqrt{3}.$
5. $\sqrt[3]{2}\cdot\sqrt[3]{16}=\sqrt[3]{32}=2\sqrt[3]{4}.$

Exercise 105

Multiply and simplify the following:

1. $3\sqrt{2}$ by $\sqrt{6}$.
2. $4\sqrt{3}$ by $3\sqrt{6}$.
3. $\frac{1}{2}\sqrt{2}$ by $8\sqrt{3}$.
4. $2\sqrt{8}$ by $3\sqrt{6}$.
5. $4\sqrt{\frac{1}{2}}$ by $9\sqrt{\frac{1}{3}}$.
6. $\sqrt[3]{2}$ by $\sqrt[3]{4}$.
7. $\sqrt{3}\cdot\sqrt{5}\cdot\sqrt{15}$.
8. $\frac{1}{3}\sqrt{\frac{3}{5}}\cdot\frac{5}{8}\sqrt{\frac{8}{2}}\cdot\sqrt{5}$.
9. $\sqrt{a}\cdot\sqrt{a^3}\cdot\sqrt{a^5}$.
10. $\sqrt{a^2b}\cdot\sqrt{ab^2}\cdot\sqrt{ab}$.
11. $a\sqrt{b}\cdot b\sqrt{a}\cdot\sqrt{x}$.
12. $\sqrt{2a}\cdot\sqrt{8a^3}\cdot\sqrt{3ab}$.
13. $\sqrt[3]{a^2}\cdot\sqrt[3]{a}\cdot\sqrt[3]{a^4}$.
14. $\sqrt[3]{3x}\cdot\sqrt[3]{9x}\cdot\sqrt[3]{5y}$.
15. $\frac{1}{2}\sqrt[3]{\frac{1}{3}}\cdot\frac{1}{3}\sqrt[3]{\frac{1}{9}}\cdot 24\sqrt[3]{3}$.
16. $\frac{1}{3}\sqrt[3]{a}\cdot\frac{1}{2}\sqrt[3]{b}\cdot\sqrt[3]{c}$.

119. Multiplication of radical polynomials. Polynomials containing terms that are surds may be multiplied as follows:

Illustrative examples.

1.
$$\begin{array}{r} 2\sqrt{3}+\sqrt{2} \\ 3\sqrt{3}+2\sqrt{2} \\ \hline 18+3\sqrt{6} \\ 4\sqrt{6}+4 \\ \hline 22+7\sqrt{6}.\ \text{Ans.} \end{array}$$

2.
$$\begin{array}{l} 3+\sqrt{5} \\ 2-\sqrt{3} \\ \hline 6+2\sqrt{5} \\ \qquad\qquad -3\sqrt{3}-\sqrt{15} \\ \hline 6+2\sqrt{5}-3\sqrt{3}-\sqrt{15}.\ \text{Ans.} \end{array}$$

Exercise 106

Perform the indicated multiplications:

1. $(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})$.
2. $(6+3\sqrt{2})(5-2\sqrt{2})$.
3. $(3\sqrt{5}+a)(2\sqrt{5}-3a)$.
4. $(3-\sqrt{5})(2-3\sqrt{5})$.
5. $(a+\sqrt{b})(b+\sqrt{a})$.
6. $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$.
7. $(2\sqrt{5}-\sqrt{3})(2\sqrt{5}+\sqrt{3})$.
8. $(3\sqrt{6}-2\sqrt{5})(3\sqrt{6}+2\sqrt{5})$.

9. $(6-\sqrt{5})(6+\sqrt{5})$.

10. $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$.

11. $(\sqrt{2}+\sqrt{3}-\sqrt{5})(\sqrt{2}-\sqrt{3}+\sqrt{5})$.

12. $(2\sqrt{3}+\sqrt{5}-3\sqrt{2})(3\sqrt{3}+2\sqrt{5}+2\sqrt{2})$.

13. $(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}-\sqrt{b}-\sqrt{c})$.

14. $(2\sqrt{3}+\sqrt{5})^2$. 15. $(5\sqrt{a}-6\sqrt{b})^2$.

120. Division of radicals. The process of division of radical expressions is accomplished by placing the dividend and divisor in fractional form and multiplying numerator and denominator of the resulting fraction by such a number as will make the denominator a rational number.

As, for example, if we multiply both terms of the fraction $\frac{\sqrt{3}}{\sqrt{2}}$ by $\sqrt{2}$, the result becomes $\frac{\sqrt{6}}{2}$. Explain. Has the value of the fraction been changed? If we wish to approximate $\frac{\sqrt{3}}{\sqrt{2}}$ to decimal form, will it be easier to substitute the decimal values in the original fraction or in its equivalent, $\frac{\sqrt{6}}{2}$?

The **rationalizing factor** of an irrational number is a number which, if multiplied by the original number, gives a rational result.

The rationalizing factor of $\sqrt{2}$ is $\sqrt{2}$, since $\sqrt{2}\cdot\sqrt{2}=2$. Likewise the rationalizing factor of $\sqrt{8}$ is $\sqrt{2}$, of $\sqrt[3]{4}$ is $\sqrt[3]{2}$, of $\sqrt{a^2b}$ is $\sqrt{b}$, and of $2\sqrt{3}$ is $\sqrt{3}$.

Multiply $\sqrt{x}+\sqrt{y}$ by $\sqrt{x}-\sqrt{y}$. What is the rationalizing factor of $\sqrt{x}+\sqrt{y}$? of $\sqrt{x}-\sqrt{y}$?

A binomial radical expression, which contains no surd of a greater degree than the second, has for its rationalizing factor a binomial differing from the first only in the sign of one of the terms. Two such binomial radical expressions are called **conjugate surds.**

Illustrative examples.

1. $\frac{\sqrt{5}}{\sqrt{2}}=\frac{\sqrt{5}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}=\frac{\sqrt{10}}{2}$. 2. $\frac{\sqrt{3}}{\sqrt{8}}=\frac{\sqrt{3}\cdot\sqrt{2}}{\sqrt{8}\cdot\sqrt{2}}=\frac{\sqrt{6}}{4}$.

3. $\frac{5}{\sqrt{3}-\sqrt{2}}=\frac{5(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}=\frac{5\sqrt{3}+5\sqrt{2}}{3-2}=?$

4. $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}=\frac{(\sqrt{x}+\sqrt{y})(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}=\frac{x+2\sqrt{xy}+y}{x-y}$.

Exercise 107

Simplify the following:

1. $\frac{1}{\sqrt{2}}$. 2. $\frac{3}{\sqrt{3}}$. 3. $\frac{1}{\sqrt{x}}$.

4. $\frac{\sqrt{x}}{\sqrt{y}}$. 5. $\frac{2\sqrt{a}}{a\sqrt{2}}$. 6. $\frac{\sqrt{2}}{2\sqrt{3}}$.

7. $\frac{2\sqrt{a}}{3\sqrt{b}}$. 8. $\frac{3\sqrt{2}}{2\sqrt{3}}$. 9. $\frac{\sqrt{xy}}{\sqrt{xz}}$.

10. $\frac{2}{\sqrt{5}-\sqrt{3}}$. 11. $\frac{5+\sqrt{3}}{2-\sqrt{3}}$. 12. $\frac{a}{\sqrt{a}+\sqrt{b}}$.

13. $\frac{\sqrt{a}+\sqrt{b}}{2\sqrt{a}+3\sqrt{b}}$ 14. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$. 15. $\frac{2\sqrt{5}-\sqrt{2}}{3\sqrt{5}+2\sqrt{2}}$.

121. Irrational equations. Equations containing radicals may sometimes be solved by the axiom of § 116.

Illustrative examples.

1. $\sqrt{x+5}=4$.
   Then $x+5=16$. Squaring both members.
   Whence $x=11$. Check.
2. $\sqrt{x+11}-1=\sqrt{x}$.
   $\sqrt{x+11}=\sqrt{x}+1$. Transposing.
   $x+11=x+2\sqrt{x}+1$. Squaring both members.
   Then $10=2\sqrt{x}$ or $5=\sqrt{x}$. Whence $x=25$. Check.

Exercise 108

Solve and check the following:

1. $\sqrt{x+3}=3.$
2. $\sqrt{2x-5}=3.$
3. $\sqrt{x^2+5}=x+1.$
4. $\sqrt{x^2-7}=x-1.$
5. $\sqrt{x+5}=\sqrt{2x+1}.$
6. $\sqrt{x-7}=\sqrt{x}-1.$
7. $\sqrt{x^2+11}-x=1.$
8. $2\sqrt{x+6}=\sqrt{44}.$
9. $7=x+\sqrt{x^2-7}.$
10. $\sqrt{x}-\sqrt{5}=\sqrt{x-5}.$
11. $16-\sqrt{x}=\sqrt{32+x}.$
12. $\dfrac{\sqrt{x}-3}{\sqrt{x}+3}=\dfrac{\sqrt{x}-4}{\sqrt{x}+1}.$
13. $\dfrac{\sqrt{x-5}+1}{\sqrt{x-5}-1}=3.$
14. $\sqrt[3]{5x+2}=3.$
15. $\sqrt[3]{8x-3}=-3.$
16. $\sqrt[3]{10y-13}=\sqrt[3]{7y-1}.$

Exercise 109. Review

Find the square root of:

1. $16a^6-24a^5+25a^4-20a^3+10a^2-4a+1.$
2. $10r^2-12r^3-4r+9r^4+1.$
3. $a^2+b^2+c^2-2ab+2ac-2bc.$
4. $a^2+b^2+c^2+d^2-2ab+2ac-2ad-2bc+2bd-2cd.$
5. $49a^4+121b^4-73a^2b^2-198ab^3+126a^3b.$

Find the square root of the following to the nearest hundredth:

6. 13. **7.** 5.233. **8.** 932.127. **9.** .395. **10.** .0337.

Reduce the following to simplest radical form:

11. $\sqrt{50}.$ **12.** $\sqrt{27}-\sqrt{12}.$ **13.** $\sqrt{75}+\sqrt{48}.$

14. $\sqrt{\tfrac{1}{3}}.$ **15.** $\sqrt{\tfrac{3}{5}}.$ **16.** $2\sqrt{\tfrac{1}{2}}+\sqrt{98}.$

17. $3\sqrt{\tfrac{1}{3}}+\sqrt{12}-\sqrt{27}.$ **18.** $\sqrt{a^2b}.$ **19.** $\sqrt{\dfrac{b}{a}}.$

20. $\sqrt{\dfrac{a}{b}} + \sqrt{a^3b} - a\sqrt{\dfrac{b}{a}}$ 21. $\sqrt{a^3 + 4a^2b + 4ab^2}$.

22. $\sqrt{a^3 - a^2b}$. 23. $\sqrt{a^3} + \sqrt{ab^2} + \sqrt{(a-b)^2a}$.

Perform the following multiplications and reduce the results to simplest radical form:

24. $(\sqrt{3} + \sqrt{2})(\sqrt{3} + \sqrt{2})$.

25. $(2\sqrt{5} - \sqrt{3})(3\sqrt{5} + 2\sqrt{3})$.

26. $(\sqrt{a} + \sqrt{b})(2\sqrt{a} - 3\sqrt{b})$.

27. $(a\sqrt{a} + b\sqrt{b})^2$.

28. $(\sqrt{a} + \sqrt{b} - \sqrt{c})(\sqrt{a} - \sqrt{b} + \sqrt{c})$.

29. $(5\sqrt{7} - 3\sqrt{x})(2\sqrt{7} + 5\sqrt{x})$.

30. $(2\sqrt{5} + 3\sqrt{7} - 2\sqrt{3})(3\sqrt{5} - 2\sqrt{7} - 3\sqrt{3})$.

Reduce the following to simplest radical form:

31. $\dfrac{\sqrt{2}}{\sqrt{3}}$. 32. $\dfrac{\sqrt{3}}{\sqrt{5}}$. 33. $\dfrac{3}{\sqrt{7}}$. 34. $\dfrac{2a}{\sqrt{b}}$.

35. $\dfrac{x}{\sqrt{y^2z}}$. 36. $\dfrac{3}{\sqrt{300}}$. 37. $\dfrac{4}{3\sqrt{5}}$. 38. $\dfrac{x-1}{\sqrt{x}}$.

39. $\dfrac{x}{\sqrt{x-1}}$. 40. $\dfrac{\sqrt{x^2y}}{\sqrt{xy^2}}$. 41. $\dfrac{1}{\sqrt{2}-1}$.

42. $\dfrac{5}{\sqrt{3}+\sqrt{2}}$. 43. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$. 44. $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$.

45. $\dfrac{2\sqrt{3}-\sqrt{5}}{3\sqrt{3}+\sqrt{5}}$. 46. $\dfrac{5+3\sqrt{2}}{\sqrt{2}-1}$. 47. $\dfrac{3+2\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

48. $\dfrac{a\sqrt{b}+b\sqrt{a}}{a\sqrt{b}-b\sqrt{a}}$. 49. $\dfrac{1+\sqrt{x-1}}{1-\sqrt{x-1}}$.

50. $\dfrac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}}$.

Reduce the following to simplest radical form and then approximate the results to decimals:

(Use the table of square roots.)

51. $\sqrt{128}$. **52.** $\sqrt{700}$. **53.** $\sqrt{243}$. **54.** $\sqrt{\frac{3}{7}}$.

55. $\frac{\sqrt{2}}{\sqrt{11}}$. **56.** $\sqrt{\frac{5}{13}}$. **57.** $\frac{\sqrt{3}}{\sqrt{8}}$. **58.** $\frac{\sqrt{2}}{\sqrt{5}}$.

59. $\frac{3\sqrt{2}+2\sqrt{3}}{\sqrt{3}}$. **60.** $\frac{2\sqrt{7}+\sqrt{5}}{3\sqrt{7}-2\sqrt{5}}$.

61. $4\sqrt{\frac{1}{2}}-\frac{12}{\sqrt{8}}$. **62.** $\frac{9+\sqrt{27}}{\sqrt{3}}$. **63.** $\frac{5-\sqrt{2}}{5+\sqrt{2}}$.

Solve the following equations and check:

64. $\sqrt{5n-19}=4$. **65.** $\sqrt{2x^2-1}=7$.

66. $x=8-\sqrt{x^2+16}$. **67.** $\sqrt{x^2-1}=1-x$.

68. $7-\sqrt{x}=\sqrt{x+7}$. **69.** $5\sqrt{n-3}=\sqrt{n+21}$.

70. $\frac{1}{a-1}-\frac{1}{a+1}=\frac{1}{12}$. **71.** $\frac{x^2+2}{5}-\frac{3x^2-7}{6}=\frac{11}{30}$.

72. $\frac{3}{x}-\frac{1}{2x}=\frac{x}{5}-\frac{5}{2x}$.

73. $(2n-5)^2-4(2n+3)(n-4)=9$.

74. $(a+7)^2+(a-7)^2=100$.

75. $(3a+4)^2+(a-12)^2=180$.

Solve the following equations for r:

76. $\pi r^2=A$. Ans. $r=\pm\sqrt{\frac{A}{\pi}}$.

77. $\pi r^2 a=V$. **78.** $4\pi r^2=S$.

79. $\frac{4}{3}\pi r^3=V$. Ans. $r=\sqrt[3]{\frac{3V}{4\pi}}$. **80.** $\frac{1}{3}\pi r^2 a=V$.

Solve the following problems:

81. Find the diagonal of a square whose side is 8 inches; 10 inches; 25 inches.

82. If d represents the diagonal of a square and s a side, show that $d = s\sqrt{2}$.

83. Use the formula of No. 82 to solve the three parts of No. 81.

84. Find the side of a square whose diagonal is 6 inches; 10 inches; d inches.

85. The length of a rectangle is twice the width and the area is 72 square inches. Find the dimensions and the diagonal.

86. The foot of a ladder is placed 7 feet from the wall of a house and the top just reaches a window 19 feet from the ground. What is the length of the ladder?

87. A boy wishes to determine the height of a tree. He finds by the aid of his kite string that it is 100 feet from the top of the tree to a point on the ground 80 feet from the foot of the tree. What is the tree's height?

88. Find the altitude and area of an equilateral triangle whose side is 4 inches; 20 inches; 11 inches.

89. If s is the number of inches in the side of an equilateral triangle and A is the number of square inches in the area, show that $A = \frac{s^2}{4}\sqrt{3}$.

90. Use the formula of No. 89 to solve the three parts of No. 88.

91. There is a certain number such that 9 minus its square root is equal to the square root of the sum of the number and 9. Find the number.

92. If a and b represent the lengths of the two sides of a right triangle and c the hypotenuse, show that $a^2 = (c - b)(c + b)$.

CHAPTER IX.

QUADRATIC EQUATIONS

122. Definitions. A **quadratic equation** has been defined as an equation of the second degree with respect to the unknown number. (See § 63.)

A quadratic equation that contains no term of the first degree of the unknown is called a **pure quadratic equation.**

$x^2 = 16$ is a pure quadratic equation.

A quadratic equation that contains terms of both the first and second degree of the unknown is called an **affected quadratic equation.**

$x^2 - 5x = 6$ and $x^2 = 3x$ are affected quadratic equations.

A pure quadratic equation is sometimes called an **incomplete quadratic,** and an affected quadratic, if it has one term that does not contain the unknown, is called a **complete quadratic equation.**

Every quadratic equation in x may be reduced to the form $ax^2 + bx + c = 0$, where a may have any known value other than 0, and b and c may have any known values whatsoever.

123. Review. The student has already become familiar with the solution of quadratic equations by factoring, and also with the solution of pure quadratics. (See §§ 64 and 116.) These and other necessary processes are reviewed in the following exercises.

Exercise 110

Solve by factoring and check:

1. $x^2 - 9x + 20 = 0$.
2. $x^2 + 8x = 9$.
3. $2x^2 - 5x + 2 = 0$.
4. $7x^2 = 5x$.
5. $x^2 - 9 = 0$.
6. $x^2 - x = 0$.
7. $2x^2 + x = 6$.
8. $n^3 - n = 0$.

Simplify, if necessary, to the form $x^2 = a$, and then solve by taking the square root of each member:

9. $x^2 = 20$.
10. $x^2 = 18$.
11. $3x^2 = 15$.
12. $(x + 4)^2 + (x - 4)^2 = 50$.
13. $\dfrac{x-a}{x+a} + \dfrac{x+a}{x-a} = \dfrac{10}{3}$.
14. $\dfrac{x-3}{x+3} + \dfrac{x+3}{x-3} = 8$.

Expand the following:

15. $(x - 3)^2$.
16. $(x + 5)^2$.
17. $(2n - 7)^2$.
18. $(3a - 5)^2$.
19. $(2x + 7y)^2$.
20. $(x + \frac{3}{2})^2$.
21. $(n - \frac{7}{2})^2$.
22. $(a + \frac{2}{3})^2$.
23. $\left(a + \dfrac{b}{5}\right)^2$.

Supply the missing terms in the following trinomial squares:

24. $a^2 + 6a + ?$
25. $x^2 + 10x + ?$
26. $b^2 - 12b + ?$
27. $c^2 - 20c + ?$
28. $x^2 - ?x + 36$.
29. $b^2 + ?b + 49$.
30. $a^2 - ?a + 25$.
31. $4n^2 - 12n + ?$
32. $4a^2 + ?a + 9$.
33. $4a^2 + 20a + ?$
34. $16a^2 + 24a + ?$
35. $16a^2 - 48a + ?$

36. The second of two numbers is 3 more than the first and their product is 40. Find the numbers.

37. Find three consecutive numbers the sum of whose squares is 110.

38. Find two consecutive even integers whose product is 48.

39. The page of a certain book is 2 inches longer than it is wide and its area is 35 square inches. Find its dimensions.

40. A rectangle is 3 inches longer than it is wide and its area is 108 square inches. Find its dimensions.

41. A certain number is subtracted from both numerator and denominator of the fraction $\frac{8}{11}$. The result would have been the same if the number had been added to the denominator. Find the number.

42. The sum of a number and its square is 56. What is the number?

43. One side of a right triangle is 5 inches longer than the other side and the hypotenuse is 25 inches. Find the two sides.

44. The sum of two numbers is 25 and their product is 150. Find the numbers.

45. One number is 3 more than another. The square of the sum of the two numbers exceeds the sum of their squares by 80. What are the numbers?

124. Solution of the quadratic equation by completing the square.

First method. When the coefficient of x^2 is 1.

A study of the following **illustrative examples** will help to make clear the steps of the process.

I. Solve $x^2 + 6x = 3$.

(1) $x^2 + 6x + 9 = 12$. (Make the left member a perfect square by adding 9 to each member.)

(2) $x + 3 = \pm 3.464$. [Extracting the square root of each member of (1).]

(3) Therefore $x = \pm 3.464 - 3$. And $x = .464$ or -6.464.

II. Solve $12x = x^2 + 32$.

(1) $x^2 - 12x = -32$. (Explain.)

(2) $x^2 - 12x + 36 = 4$. (Explain.)

(3) $x - 6 = \pm 2$. Find the values of x.

Observations. (1) The equation must be changed to the form $x^2 + bx = c$. That is, the coefficient of x^2 must be $+1$, and both terms containing x must be in the left member.

(2) The number to be added to each member of the equation is the square of half the coefficient of x.

(3) The square root of the second member must have the $\pm$ sign.

Exercise 111

Solve the following equations by completing the square:
(Use $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, and $\sqrt{5} = 2.236$.)

1. $x^2 + 6x = 7$.
2. $n^2 - 10n = 24$.
3. $a^2 + 8a = -7$.
4. $x^2 - 12x = 28$.
5. $m^2 + 18m = -17$.
6. $R^2 - 6R = 112$.
7. $x^2 + 2x = 1$. Ans. $x = .414$ and -2.414.
8. $x^2 - 4x = 1$.
9. $a^2 - 8a = -14$.
10. $c^2 - 10c = -22$.
11. $R^2 - 2R = 4$.
12. $m^2 - 6m = -4$.
13. $x^2 - 20x = -97$.
14. $x^2 + 3x = 4$.

Solution. $x^2 + 3x + \frac{9}{4} = \frac{25}{4}$. (Explain.)
Whence $x + \frac{3}{2} = \pm \frac{5}{2}$. Therefore $x = 1$, or $x = -4$.

15. $x^2 + 5x = -4$.
16. $x^2 - 5x = 6$.
17. $a^2 - 7a = 8$.
18. $n^2 + 3n = 10$.
19. $b^2 - 7b + 10 = 0$.
20. $c^2 - 7c = 30$.
21. $R^2 - R = 20$.
22. $m^2 + 3m - 28 = 0$.
23. $W^2 - 11W = 60$.
24. $S^2 + 5S = 36$.
25. $3n^2 - 12n + 9 = 0$.

Suggestion. First divide both members by 3.

26. $5x^2 + 10x = 20$.
27. $7a^2 - 28a + 7 = 0$.
28. $4x^2 - 16x + 8 = 0$.
29. $3b^2 - 18b + 21 = 0$.
30. $6m^2 - 24m + 18 = 0$.
31. $11W^2 - 44W - 55 = 0$.
32. $5x^2 - 15x = 20$.
33. $3R^2 + 9R = -6$.
34. $7x^2 - 63x = -56$.
35. $3B^2 - 12B = 36$.
36. $2R^2 - 6R = 80$.
37. $3Y^2 - 18Y = 48$.

38. $m^2 + 5m + 2 = 0.$

Solution. $m^2 + 5m + \frac{25}{4} = \frac{25}{4} - 2 = \frac{17}{4}$. **Explain.**

Whence $m + \frac{5}{2} = \pm\sqrt{\frac{17}{4}} = \pm\frac{4.123}{2}$ making $m = -4.561$, or $m = -.4385$.

(Notice that we have found two factors of 2 whose sum is 5, or the factors of $m^2 + 5m + 2$ are, approximately, $m + .4385$ and $m + 4.5615$. Test by multiplying.)

39. $a^2 + 7a + 5 = 0.$

40. $m^2 - 9m = 5.$

41. $n^2 - 11n = 3.$

42. $x^2 - 9x = -4.$

43. $3x^2 + 5x = -2.$

Solution. $x^2 + \frac{5}{3}x = -\frac{2}{3}$

Whence $x^2 + \frac{5}{3}x + \frac{25}{36} = \frac{25}{36} - \frac{2}{3} = \frac{1}{36}$ and $x + \frac{5}{6} = \pm\frac{1}{6}$.

Therefore $x = -1$, or $x = -\frac{2}{3}$.

44. $2x^2 - 5x + 2 = 0.$

45. $5x^2 + 9x + 4 = 0.$

46. $5x^2 - 11x = 5.$

47. $3x^2 + 7x = -2.$

The Hindoo method.

If we are to solve the equation $4x^2 - 12x + 9 = 25$, it is evident that the square root of both members can be extracted at once, giving $2x - 3 = \pm 5$, whence $x = 4$ or $x = -1$. This avoids the fractions which would have entered the left member if the first method for completing the square had been followed.

Illustrative example:

Solve $3x^2 - 5x = 2$. (1)

$36x^2 - 60x = 24$. (2) (Multiplying both members by 12.)

$36x^2 - 60x + 25 = 49$. (3) (Completing the square.)

$6x - 5 = \pm 7$. (4) (Extracting the square root of each member.)

Whence $x = 2$, or $-\frac{1}{3}$.

Observations. (1) The equation should be put in the simplest possible form with the known number in the right member.

(2) Multiply both members of the equation by four times the coefficient of x^2.

(3) Divide the second term of (2) by twice the square root of the first term and square the result to determine the number that completes the square. Notice that the number to be squared is the coefficient of x in (1).

(4) Extract the square root of each member of (3) and solve the resulting equations.

Exercise 112

Solve the folowing equations by the Hindoo method:

1. $3x^2 - 5x = -2$.
2. $4n^2 - 3n = 7$.
3. $a^2 + 3a = 10$.
4. $2x^2 - 7x = -6$.
5. $3n^2 - 5n = 8$.
6. $3n^2 = 2 + n$.
7. $x(x - 1) = 6$.
8. $3x^2 + 2x = 3$. Ans. $x = .721$ or -1.387.
9. $3x^2 + 5x + 1 = 0$.
10. $3x^2 - 5x + 1 = 0$.
11. $2n^2 - 5n = -2$.
12. $3n - 2(n - 1)(n + 1) = 0$.
13. $3n^2 - 4n = 7$.

Note. Multiply both members of No. 13 by 3 instead of by 12. It will be observed that when the coefficient of the second term is an even number, fractions will be avoided by multiplying both members by the coefficient of x^2. Use this modified form of the Hindoo method wherever possible.

14. $3x^2 - 10x + 3 = 0$.
15. $3x^2 + 2x = 2$.
16. $5x^2 - 4x = 9$.
17. $5x^2 + 4x = 9$.
18. $2x^2 + 2x = 1$.
19. $3n^2 - 4n = 3$.
20. $3a^2 + 8a = 3$.
21. $2x^2 - 5ax + 3a^2 = 0$. (Solve for x.)
22. $6n^2 - 7an + 2a^2 = 0$. (Solve for n.)
23. $8n^2 - 3an = 11a^2$.

Solve the following equations by either process:

24. $11n = n^2 + 30$.
25. $x^2 + 60 = 16x$.
26. $2x^2 + 7x = 15$.
27. $(2x - 3)^2 = (x + 2)^2$.
28. $x(x + 1) = 3$.
29. $3x^2 + 5x = 0$.
30. $4x^2 = 7x$.
31. $3(2x - 1)^2 + 3(x + 1)^2 = 10(2x - 1)(x + 1)$.

125. Solution of quadratic equations by formula. It was shown in § 122 that every quadratic equation in x may be reduced to the form $ax^2 + bx + c = 0$. Since this equation represents any quadratic equation whatsoever, it is evident that if we solve this equation for x, the resulting roots may be used as formulas to get the roots of any given quadratic. Study the following:

(1) $ax^2 + bx + c = 0$. Then $ax^2 + bx = -c$.

(2) $x^2 + \frac{bx}{a} = \frac{-c}{a}$. (Dividing both members by a.)

(3) $x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$. (Completing the square.)

(4) $x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$. (Extracting the square root of each member.)

(5) Then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

If we wish to solve a given quadratic equation by the formula, we have only to take the numbers that are used in place of a, b, and c in the given equation and substitute these numbers for a, b, and c in the formula. As, for example,

$3x^2 - 5x + 2 = 0$. Here $a = 3$, $b = -5$, and $c = 2$.

Then $x = \frac{5 \pm \sqrt{25 - 24}}{6}$. $x = 1$, or $x = \frac{2}{3}$. Explain.

Observations. (1) All the terms of the equation must be collected in the left member.

(2) The first term, or the term containing x^2, should be positive. (Multiply all terms by -1, if necessary.)

(3) Substitute the numbers that correspond to a, b, and c and reduce the result to simplest form. Approximate to decimals if necessary.

Exercise 113

Solve the following equations, using the formula:

1. $x^2 - 5x + 6 = 0$.

2. $2x^2 + 3x + 1 = 0$.

3. $2x^2 + 3x - 2 = 0$.

4. $3x^2 + 5x = 2$.

5. $2x^2 - 4x + 1 = 0$.

6. $3x^2 + 4x = 7$.

7. $3x^2 - 8x - 4 = 0$.

8. $3x^2 - 6 = 7x$.

9. $27 = 2x^2 - 3x$.
10. $x^2 - 2x - 1 = 0$.
11. $8 = x(x - 4)$.
12. $x^2 = 3(1 - 2x)$.
13. $4x = (4 - x)(4 + x)$.
14. $(x + 1)^2 + (x + 2)^2 = 85$.
15. $5x^2 + 31x + 6 = 0$.
16. $3x - 1 = - 5x^2$.
17. $x(x - 1) = 4$.
18. $7 = 2x(x - 2)$.
19. $6x(x - 2) = - 2$.
20. $3x^2 - 5x = 8$.

126. Simultaneous equations. The student has already met and solved pairs of simultaneous equations, one linear and one quadratic. (See § 105.) No new principles are involved in the following exercises.

Exercise 114

Find all sets of roots that satisfy the following pairs of simultaneous equations and check:

1. $x + y = 5$
   $xy = 6$.
2. $x + 2y = 7$
   $2xy = 12$.
3. $x + 3y = 10$
   $xy = 8$.
4. $x^2 + y^2 = 13$
   $x + y = 5$.
5. $x^2 - y^2 = 11$
   $x + y = 11$.
6. $x^2 + y^2 = 18$
   $x - y = 6$.
7. $x^2 + y^2 = 11$ Ans. $x = 3.225$ when $y = .775$, and
   $x + y = 4$. $x = .775$ when $y = 3.225$.

Check these roots carefully. Do they check exactly? Why?

8. $2x^2 + y^2 = 5$ Ans. $x = 1.549$ and $-.216$
   $x + y = 2$. $y = .451$ and 2.216.
9. $2x^2 + 3y^2 = 16$
   $x + y = 3$.
10. $\frac{x}{y} + \frac{y}{x} = \frac{5}{2}$
    $x + y = 3$.
11. $x^2 + xy + y^2 = 19$
    $x + y = 5$.
12. $x = 6 - 6y$
    $y^2 = x + 10$.

13. $x^2 + y^2 = 74$
$x - y = 2.$

14. $x^2 - y^2 = 72$
$x - y = 6.$

15. $9x^2 - 4y^2 = 29$
$3x + 2y = 29.$

16. $x^2 + 2y^2 = 17$
$x + 2y = 7.$

17. $x^2 = y$
$x + y = 6.$

18. $x^2 + 2x + 1 = 2y^2$
$x + y = 8.$

19. $3x^2 - 4y^2 = 11$ (1)
$2x^2 + 3y^2 = 30.$ (2)

Solution. $6x^2 - 8y^2 = 22$ (3) Multiplying (1) by 2.
$6x^2 + 9y^2 = 90$ (4) Multiplying (2) by 3.
$-17y^2 = -68$ Subtracting (4) from (3).

Whence $y = 2$, and $y = -2$. Explain and find x.

There are four sets of answers for the above pair of equations. Check each set.

20. $5x^2 - y^2 = 1$
$2x^2 + 7y^2 = 30.$

21. $5x^2 - 7y = -15$
$3x^2 + 8y = 52.$

22. $3x^2 - 4y^2 = 11$
$4x^2 - 7y^2 = 8.$

23. $3x^2 - 20y^2 = -5$
$4x^2 - 24y^2 = 4.$

24. $x^2 + y^2 = 26$ (1)
$xy = 5.$ (2)

Solution. $2xy = 10.$ (3) Multiplying (2) by 2.
$x^2 + 2xy + y^2 = 36.$ (4) Adding (1) and (3).
Whence $x + y = \pm 6.$ (5) Explain.
$x^2 - 2xy + y^2 = 16.$ (6) Subtracting (3) from (1).
$x - y = \pm 4.$ (7)

There are four ways of combining the linear equations (5) and (7), which gives four sets of roots. Find them and check.

25. $x^2 + 4y^2 = 25$
$xy = 6.$

27. $x^2 + y^2 = 53$
$xy = 14.$

26. $x^2 + y^2 = 5$
$xy = 2.$

28. $x^2 + y^2 = 65$
$xy = 8.$

127. Graphing quadratic equations. The student has already learned how to graph an equation of the first degree in two unknowns and has discovered that the graph of such an equation is a straight line. (See § 84.)

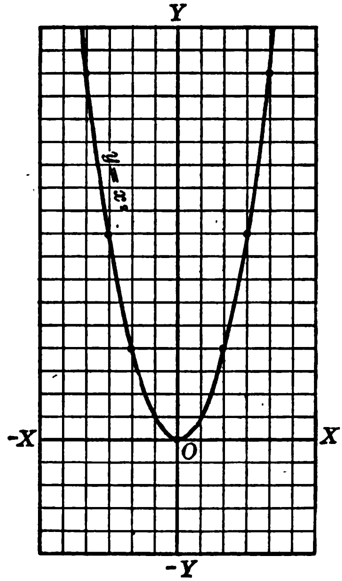
The graph of a quadratic equation, however, is not a straight line, and it will be necessary to find many sets of values that satisfy the equation, if we expect good results.

I. Graph $y = x^2$.

Substituting several different values for x, both positive and negative, we obtain the following sets:

$x =0,$	$1,$	$-1,$	$2,$	$-2,$	$3,$	$-3,$	$4,$	-4
$y =0,$	$1,$	$1,$	$4,$	$4,$	$9,$	$9,$	$16,$	16

Find these points on the cross-section paper and draw a smooth curve through them. This particular curve is a **parabola.**



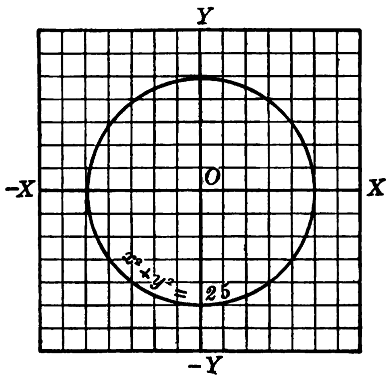
II. Graph $x^2 + y^2 = 25$.

$y = \pm\sqrt{25 - x^2}$. Explain.

Substituting as before

$x = 0,$	$0,$	$3,$	$3,$	$-3,$	$-3,$	$4,$	$4,$	$-4,$	-4
$y = 5,$	$-5,$	$4,$	$-4,$	$4,$	$-4,$	$3,$	$-3,$	$3,$	-3

Note. No number should be substituted for x that will make $25 - x^2$ negative, since we cannot take the square root of a negative number. We must keep in mind that every time we take the square root of a number, we get two results, one positive and the other negative. If the number whose root is wanted is not a perfect square, the results should be approximated to the nearest tenth.

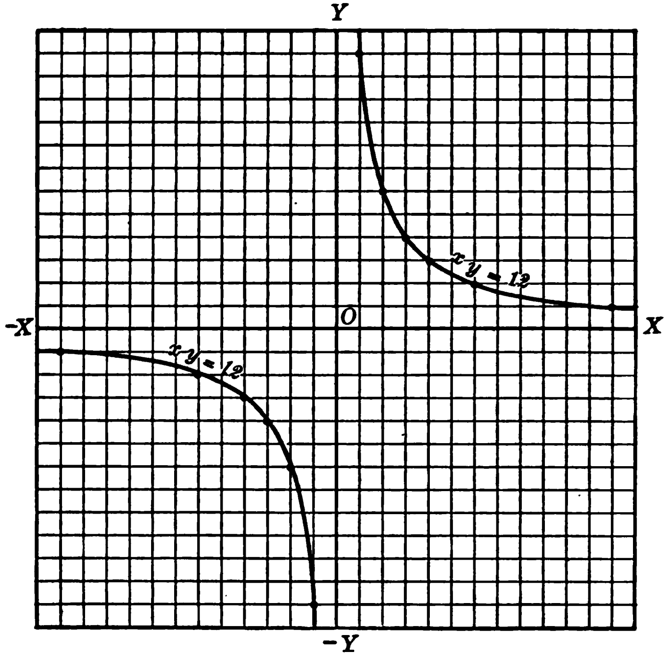


Find the above points and draw a curved line through them. We should get a circle with a radius of five units.

III. Graph $xy = 12$. $y = \frac{12}{x}$. Why?

$x = 1, -1, 2, -2, 3, -3,$ $x = 4, -4, 6, -6, 12, -12.$
$y = 12, -12, 6, -6, 4, -4,$ $y = 3, -3, 2, -2, 1, -1.$

Note. This is a curve of two parts or branches and is called a **hyberbola.**



Exercise 115

Graph the following:

1. $y = x^2 + 3$. 2. $y = x^2 + 2x + 1$. 3. $x^2 + y^2 = 36$.
4. $x = y^2$. (Substitute values for y.) 5. $x = y^2 - 4$.
6. $y = -x^2$.
7. $4x^2 + 9y^2 = 36$. (This curve is an **ellipse.**)
8. $9x^2 + 4y^2 = 36$. 9. $x^2 - y^2 = 9$.

Graph on the same axes each of the following pairs of equations and write the values of the points of intersection:

10. $x^2 + y^2 = 25$ $x - y = 1.$	**11.** $x + y = 7$ $xy = 10.$
12. $9x^2 - 4y^2 = 36$ $3x + 2y = 6.$	**13.** $x^2 - y^2 = 16$ $x + y = 8.$
14. $3x^2 + 2y^2 = 45$ $x - y = 0.$	**15.** $x + y = 0$ $xy = -10.$

Exercise 116. Problems

Solve the following problems using either one or two unknowns. Any roots of the equations that do not satisfy the conditions of the problem must be omitted.

1. Find two numbers whose sum is 3, if the sum of their squares is 29.

2. Find two numbers such that the sum of the first and 2 times the second is 7 and the sum of the squares of the two numbers is 13.

3. The perimeter of a rectangle is 20 inches and the area is 24 square inches. Find the length and breadth of the rectangle.

4. The perimeter of a rectangle is 28 inches and the length of the diagonal is 10 inches. Find the dimensions.

5. The sum of two numbers added to their product is 11. Find the numbers if their difference is 1.

6. The sum of the squares of two numbers is 61 and the difference of their squares is 11. What are the numbers?

7. Find two numbers such that 3 times the square of the first plus 2 times the square of the second is 30, and 5 times the square of the first minus 3 times the square of the second is -7.

8. The sum of two numbers is 5 and the sum of their cubes is 35. What are the numbers?

9. The difference of two numbers is 1, and the difference of their cubes is 7. What are the numbers?

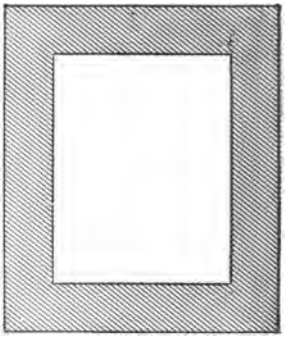
10. The area of a rectangle is 60 square feet and the length is 2 feet more than twice the width. Find the length and width.

11. The diagonal of a rectangle is 5 feet and the area is 12 square feet. Find its dimensions.

12. The area of a rectangle is 40 square feet and its perimeter is 26 feet. Find the dimensions.

13. One of two numbers is 6 larger than the other and their product is 39 larger than their sum. Find the numbers.

14. A picture without the frame is 3 inches longer than wide. The frame is $1\frac{1}{2}$ inches wide, and the area of picture and frame is double the area of the picture alone. Find the width and length of the picture.



15. The larger of two numbers divided by the smaller is equal to the quotient of the smaller divided by the larger minus 5. Find the numbers if their sum is 15.

16. The sum of two numbers is 5 and their product is 1 more than the difference of their squares. What are the numbers?

17. The number of note-books that can be bought for \$1.50 is 5 less than the number of cents that one note-book costs. Find the number of note-books and the cost of one.

18. If the product of two numbers is multiplied by their sum, the result is 70. Find the numbers if their product is 10.

19. The difference of two numbers is 7 and the difference of their squares is 91. What are the numbers?

20. Mrs. Brown buys a certain number of oranges for one dollar. If each orange had cost 1 cent less, she would have received 5 more oranges for the dollar. How many oranges did she get and how much did each orange cost?

21. The sum of the numerator and denominator of a certain fraction is 5. If the numerator is multiplied by 4 and the denominator by 9, the value of the resulting fraction is equal to the original fraction inverted. Find the fraction.

22. If the product of two numbers is added to their difference, the sum is 7. Find the numbers if the larger is 1 more than the smaller.

23. The sum of a number and its reciprocal is $\frac{5}{2}$. Find the number.

24. A string is stretched from the southeast corner of the floor of a room to the northwest corner of the ceiling. The room is 9 by 12 feet and 8 feet high. Find the length of the string.

Suggestion. First find the diagonal of the floor.

CHAPTER X

RATIO AND PROPORTION

128. Definitions. Define the following terms (see § 79): ratio, antecedent, consequent, proportion, terms of a proportion, extremes, means.

In the proportion $\frac{a}{b}=\frac{c}{d}$, state which are antecedents, consequents, means, extremes.

If the means of a proportion are the same as in $\frac{a}{x}=\frac{x}{b}$, the common mean is called the **mean proportional** between the other two terms and b is known as the **third proportional** to a and x since there are only three distinct numbers in the proportion, a, x, and b.

129. Fundamental laws of proportion.

I. *In any proportion the product of the means equals the product of the extremes.* (See § 79.)

That is, if $\frac{a}{b}=\frac{c}{d}$ then $ad=bc$. Explain how obtained.

II. *If the product of two factors equals the product of two other factors, either two may be made the extremes and the other two the means of a proportion.*

That is if $mn=xy$, then $\frac{m}{x}=\frac{y}{n}$.

Dividing both members of the equation $mn=xy$ by nx gives $\frac{mn}{nx}=\frac{xy}{nx}$, or $\frac{m}{x}=\frac{y}{n}$.

Exercise 117

Apply the first fundamental law of proportion to the solution of the following proportions and then find the values of the literal numbers:

1. $\frac{x}{2} = \frac{x+5}{4}$. 2. $\frac{x}{4} = \frac{9}{x}$. 3. $\frac{n+3}{n+5} = \frac{n-1}{n}$.

4. $\frac{n-3}{n+2} = \frac{n+3}{n+18}$. 5. $\frac{x+3}{x+7} = \frac{x-1}{x+1}$.

6. $\frac{a+1}{2a+1} = \frac{3a-2}{5a-2}$. 7. $\frac{a^2}{5a-3} = \frac{1}{2}$.

8. Given $mn = xy$, show that $\frac{m}{y} = \frac{x}{n}$.

Suggestion. Divide both members of $mn = xy$ by ny.

9. Given $xy = mn$, show that $\frac{x}{m} = \frac{n}{y}$.

Note. Numbers 8 and 9 are proofs of the second fundamental law of proportion. There are eight different proportions that can be obtained from $mn = xy$ by Law II. Write the other six.

10. Since $3 \cdot 8 = 4 \cdot 6$, apply Law II, $\frac{3}{6} = \frac{4}{8}$, $\frac{8}{6} = \frac{4}{3}$, etc. Check these two proportions by Law I and then write six different proportions from the same equality.

11. Since $2 \cdot 18 = 4 \cdot 9$, write the eight different proportions and check each by applying Law I.

12. Find two numbers in the ratio of 2 to 3, the sum of whose squares is 52.

Suggestion. Use x and y for the numbers. The equations are $\frac{x}{y} = \frac{2}{3}$, and $x^2 + y^2 = 52$.

13. Find two numbers in the ratio of 5 to 7 whose sum is 60.

14. What number must be added to each of the numbers 7, 9, 2, and 3 to give sums that form a proportion in the same order.

15. What number must be subtracted from each of the numbers 9, 10, 12, and 14 to give remainders that are in proportion?

16. 3, 5, and 9 are the first three terms of a proportion. Find the fourth term.

17. What number must be added to each term of the ratio $\frac{5}{7}$ to make the result equal to the ratio of $\frac{5}{6}$?

18. What number must be added to each term of the ratio $\frac{15}{16}$ to make the result equal to the ratio $\frac{5}{6}$? Explain your answer.

19. Find two numbers in the ratio of $\frac{3}{2}$ whose difference is 25.

20. Divide 75 into two parts that are in the ratio of 7 to 8.

21. A father and son contract to do a piece of work and agree to divide their earning in the ratio of 5 to 3. They earn \$40. How much does each receive?

22. What number must be subtracted from each term of the ratio $\frac{9}{11}$ to make the remainder equal to the ratio $\frac{1}{2}$?

23. The ratio of A's age to B's age is $\frac{3}{4}$. In 10 years the ratio of their ages will be $\frac{4}{5}$. Find their ages now.

24. A and B invest \$3000 and \$5000 in a business and agree to divide their earnings in the ratio of their investments. If the earnings for the first year are \$1000, what is the share that each receives?

25. The denominator of a certain fraction is 4 more than its numerator. If 10 is added to the numerator, the new fraction will be equal to the old fraction inverted. What is the fraction?

130. Problems involving the mean or the third proportional.

To find the mean proportional between two numbers

such as a and b it is necessary to solve the proportion $\frac{a}{x}=\frac{x}{b}$.

Applying Law I, $x^2=ab$ and $x=\pm\sqrt{ab}$. This gives the

Rule. *The mean proportional between two numbers is the square root of their product.*

Similarly, to find the third proportional to two numbers, a and b, requires the solution of the proportion $\frac{a}{b}=\frac{b}{x}$.

Solving gives $x=\frac{b^2}{a}$. This gives the following

Rule. *The third proportional to two numbers is the square of the second divided by the first.*

Exercise 118

1. Find the following mean proportionals: between 2 and 8; between 3 and 27; between 4 and 9; between 2 and 18; between 5 and 20; between 2 and 50; between 8 and 18; between 9 and 16.

2. State as a radical in simplest form the mean proportional between 2 and 4; between 2 and 12; between 3 and 9; between 4 and 12; between 5 and 15; between 7 and 14.

3. Find the following mean proportionals correct to .01: between 4 and 24; between 3 and 15; between 8 and 24.

4. What is the third proportional to 2 and 4? Ans. 8.

5. What is the third proportional to 4 and 8? to 3 and 6? to 5 and 15? to 6 and 18? to 3 and 12?

6. State the following third proportionals as simple fractions,—to 5 and 8; to 6 and 8; to 3 and 8; to 7 and 9.

7. Find the mean proportional between a^3b and ab^3; between x^3y^2z and xy^2z^3; between $(a-b)^2$ and $(a+b)^2$.

131. Problems in proportion that arise from geometry. The word **similar** in geometry has the general meaning of, "having the same shape."

All circles are similar figures.

If two rectangles are similar, the ratio of their lengths is the same as the ratio of their widths.

All squares are similar figures.

In similar figures **corresponding sides,** or lines, are lines that have the same position with respect to the rest of the lines and angles of the figure.

In similar triangles the ratio of corresponding bases is the same as the ratio of corresponding altitudes, or of a pair of corresponding sides.

The ratio of the areas of similar figures is the same as the ratio of the squares of two corresponding sides.

The ratio of the volumes of two similar solids is the same as the ratio of the cubes of two corresponding edges or other linear dimensions.

Exercise 119

1. A certain rectangle is 5 inches wide and 8 inches long. The width of a similar rectangle is 10 inches. Find its length.

Suggestion. If x is the number of inches in its length, the equation is $\frac{5}{10} = \frac{8}{x}$.

2. Find the diagonals of the rectangles in No. 1 and their ratio.

3. The sides of two squares are respectively 4 inches and 6 inches. Find their diagonals and the ratio of the diagonals.

4. If a square is equal in area to a given rectangle, the

side of the square is a mean proportional between the width and the length of the rectangle. Show this to be true.

($s^2 = ab$ and apply Law II.)

5. The bases of two similar triangles are 7 inches and 11 inches. If the altitude of the first is 14 inches, what is the altitude of the second?

6. The shadow of a certain tree is 42 feet long, and at the same time the shadow of a 5 foot pole is 3 feet in length. Find the height of the tree.

7. The three sides of a triangle are 6, 8, and 12 inches. If the shortest side of a similar triangle is 9 inches, what are the other two sides?

8. The bases of two similar triangles are 5 inches and 8 inches. If the area of the smaller is 50 square inches, what is the area of the larger?

Suggestion. Since the ratio of the areas is the same as the ratio of the squares of corresponding sides, the equation is $\frac{25}{64} = \frac{50}{x}$.

9. If one rectangle is 5 inches by 8 inches and another 10 inches by 16 inches, are they similar? What is the ratio of their areas? of their diagonals?

10. The widths of two similar rectangles are 6 inches and 10 inches, respectively. If the area of the first is 54 square inches, find the length of the second rectangle.

11. Two circles have radii of 4 inches and 5 inches, respectively, What is the ratio of their circumferences? of their areas?

12. If the radius of one circle is twice that of another, what is the ratio of their areas? of their circumferences?

13. If the area of a square is 16 square inches, what is the area of a square whose side is twice that of the first? one-half?

14. What is the ratio of the sides of two squares if the ratio of their areas is 1 to 2? 1 to 3? 2 to 5?

15. What is the ratio of the radii of two circles if the ratio of their areas is 1 to 2? 1 to 3? 2 to 5?

16. Two boxes are similar and each dimension of the larger is twice the corresponding dimension of the smaller. What is the ratio of their volumes? of their corresponding surfaces?

17. If the diameter of the sun is 110 times that of the earth, what is the ratio of their volumes? of their surfaces?

18. Find the respective diameters of a baseball and a basketball and determine the ratio of their surfaces; of their volumes.

19. If the diameters of two soap bubbles are 2 inches and 3 inches what is the ratio of their volumes?

20. A quarter-section of land contains 160 acres. If it is in the shape of a square, what is the length of one side? What is the length of a 40-acre field in the form of a square?

21. A farmer has two granaries exactly alike in all respects, except that the dimensions of one are 10 per cent greater than those of the other. How do their volumes compare? If the smaller holds 800 bushels of wheat, what is the capacity of the larger?

22. Two rectangles are similar and the base of one is 2 inches longer than the base of the other. The ratio of their areas is $\frac{9}{16}$. Find their bases.

Suggestion. Use the equations, $x = y - 2$ and $\frac{x^2}{y^2} = \frac{9}{16}$.

23. Show that the dimensions of a can that holds one gallon are just twice the dimensions of a can of similar shape that holds one pint.

SECOND COURSE IN ALGEBRA

CHAPTER XI

ESSENTIAL REVIEWS

132. Symbols. Brevity of expression is obtained in mathematics by the use of symbols. In general, such symbols may be classified according as they are used—

(1) to represent number or quantity,

(2) to indicate operations and establish relations.

In arithmetic, the digits of the so-called Arabic notation (more properly called the Hindoo notation, for the characters and the place system were probably invented by the Hindoos), 1, 2, . . . 9, stand for particular and definite numbers. Such numbers may be known as **arithmetical, Arabic,** or **Hindoo numbers.** Similarly in the old Roman notation the capital letters were used to represent particular numbers.

The simple formulas used in arithmetic, such as $A=lw$, $i=prt$, and $C=2\pi r$, introduced to the student the idea of representing general numbers by the use of the letters of the English and other alphabets. Numbers represented by letters are known as **literal numbers.**

It is often convenient to represent distinct numbers of the same general group by the use of subscripts, such as r_1 and r_2 (read "r sub 1 and r sub 2") for the radii of two circles.

Exercise. Let the student make a list of all the symbols of operation and relation that he has met so far in his study of mathematics, naming each and illustrating its use.

133. Definitions. An **algebraic expression,** or simply an **expression,** is any combination of symbols that represents a number or operations on numbers. The following are expressions: $2ab$, $a+b$, $x^3+3x^2y+3xy^2+y^3$, and $\frac{3a^2by}{7bxyz}$.

A **term** is an expression not separated by the signs + or − and representing a single number or quantity. The following are terms: $2ax$, $36x^2y$, $\frac{3a^2by}{7xyz}$, and $(a+b)$ which we read as "the quantity $a+b$."

Like or **similar terms** are terms that have the same literal parts, such as ab, $2ab$, and $7ab$, or $3a^2x$, $12a^2x$, and $77a^2x$, or $2(a+b)$, $5(a+b)$, and $25(a+b)$.

Define and write an illustration for each of the following: exponent, coefficient, monomial, binomial, trinomial, polynomial, equation of condition, identity.

In such a term as $21abxy$, 21 is called the **arithmetical** or **numerical coefficient** of the literal part, $abxy$. It should be recalled that ab is the coefficient of $21xy$ and that $21y$ is the coefficient of abx.

When we wish to represent the idea that two numbers are to be considered as opposite in character or direction, we make use of the signs + and −, and call such numbers **positive** and **negative numbers, directed numbers,** or + and − **numbers.** (See Chapter II.)

134. The following rules from Chapters II and III need emphasis.

Rule for the sign of the product of several factors. *The sign of a product is + if an even number of its factors are − and the sign of a product is − if an odd number of its factors are −.* (See §§ 30 and 39.)

Rule for the sign of a quotient. *If the dividend and divisor have like signs, the sign of the quotient is +. If they have unlike signs, the sign of the quotient is −.* (See § 31.)

Rules for the − sign with the signs of aggregation. *If an expression is placed within a sign of aggregation preceded by the − sign, the sign of each term of the expression must be changed.* (See § 37.)

If an expression is removed from a sign of aggregation preceded by the − sign, multiply each term by the coefficient before the sign of aggregation (if there is one) and change the sign of each term of the product. (See § 36.)

If an expression is placed within or removed from a sign of aggregation preceded by a + sign, no changes of signs are necessary.

Rule for exponents in multiplication. *The exponent of each literal number in a product of monomials is equal to the sum of the exponents of that number in the factors.*

It is convenient to remember this rule in its **typeforms** $a^m \cdot a^n \equiv a^{m+n}$ when a single literal number is involved and $(a^m\, b^n)\,(a^x\, b^y) \equiv a^{m+x}\, b^{n+y}$ when two or more numbers are involved. (See § 40.)

Rule for exponents in division. *The exponent of each literal number in a quotient of monomials is equal to its exponent in the dividend minus its exponent in the divisor.* (See § 41.)

The **typeforms** are $a^m \div a^n \equiv a^{m-n}$
and $a^m b^n \div a^x b^y \equiv a^{m-x}\, b^{n-y}$.

Evidently these rules apply when exponents are used with arithmetical numbers for $7^3 \cdot 7^2 = 7^5$ and $7^5 \div 7^3 = 7^2$.

Exercise 120

1. Write the sum, the difference, the product, and the quotient of the two literal numbers a and b; of $3x$ and $2y$; of abc and xyz.

2. Write the quotient when the product of x and y is divided by the sum of a and b.

3. If $a=2$, $b=3$, and $c=4$, find the numerical value of $3a^2b$; of a^3+b^2-c; of a^3+b^3; of $a^3+3a^2b+3ab^2+b^3$.

4. Give the rule for finding the sum of several similar algebraic terms, some positive and some negative.

5. Give the rule for finding the sum of several polynomials that have similar terms.

6. Give the rule for finding the difference of two polynomials that have similar terms.

7. What is the product of $(-a)(-b)(-c)$? What changes of signs will not change the sign of the product and why?

8. What is the product of $(-2)(+3)(-5)(-3)(+4)(-6)$?

9. Divide $-12a^3b^4c^5$ by $3ab^2c^3$ and state the rules used.

10. What is the product of a^2b, $-ab^2$, and a^3b^2?

11. If the quotient is b and the dividend is a, what is the divisor? What if the dividend were $-2a$?

12. If the product is x and the multiplicand y, what is the multiplier? If the product were $-2xy$?

13. Remove the parenthesis in each of the following and collect where possible: (1) $-2a-(a-b)$, (2) $a-b-(a+b-c)$, (3) $x+y-2z-(3x-2y+3z)$.

14. Express by an equation the fact that a is as much more than twice b as three times b is less than three times c.

15. Express by an equation the fact that x exceeds three times a by the quotient of y divided by twice c.

135. Fundamental processes with polynomials.

Exercise 121

1. Add $3x^3-5x^2+7x-3$, $8-x^2+x-x^3$, $7-2x^3$, $3x^2+4$, and x^3-2x^2+3x-4.

2. From the sum of $2x^2-xy+y^2$ and $x^2+3xy-y^2$, subtract the sum of $4x^2-y^2$ and $3x^2+xy+3y^2$.

3. Take the sum of $6-4a^3-a$ and $5a-1-3a^2$ from the sum of $2a^3+5-3a+2a^2$ and $3a^2-5a^3-4-a$.

4. Subtract $b-c+d-e$ from a.

5. Multiply $3x^2-5x-3$ by $2x^2+x-7$.

6. Multiply $2n^3-3n^2+4n-8$ by $3n^3+2n-3$.
7. Multiply $x^4+x^2y^2+y^4$ by $x^4-x^2y^2+y^4$.
8. Simplify $3n^2-2(n-3)\ (2n+1)$.
9. Simplify $16a^2-2(2a-3)^2-3(a-3)\ (2a+7)$.
10. Divide n^3-3n^2+3n-1 by n^2-2n+1.
11. Divide $4n^4-9n^2+30n-25$ by $2n^2-3n+5$.
12. Divide $x^2-y^2-2yz-z^2$ by $x-y-z$.
13. Divide $a^3+b^3+3abc-c^3$ by $a+b-c$.

136. Special products. If necessary, find the first product in each group of five by actual multiplication and use it as an example for obtaining the other four.

Exercise 122

I

1. $(a+b)^2$.
2. $(x+2y)^2$.
3. $(3x+5a)^2$.
4. $(2x^2+3y^2)^2$.
5. $(3a^2b^2+5x^2y^2)^2$.

II

1. $(a-b)^2$.
2. $(3x-7a)^2$.
3. $(2a^2-3ab)^2$.
4. $(x^m-y^n)^2$.
5. $(3x^a-5y^b)^2$.

III

1. $(a-b)\ (a+b)$.
2. $(3x-2y)\ (3x+2y)$.
3. $(5a^2-7b^3)\ (5a^2+7b^3)$.
4. $(x^m-y^n)\ (x^m+y^n)$.
5. $[(m-n)-7]\ [(m-n)+7]$.

IV

1. $(a+2)\ (a+3)$.
2. $(x-5)\ (x-8)$.
3. $(x-3)\ (x+7)$.
4. $(x^2-3y)\ (x^2+7y)$.
5. $[(m-n)-8]\ [(m-n)+3]$.

V

1. $(3a+2)\ (2a+3)$.
2. $(2x+3y)\ (3x+5y)$.
3. $(2a+3b)\ (3a-5b)$.
4. $(5m-3n)\ (2m+n)$.
5. $(x^2+7a)\ (3x^2-5a)$.

VI

1. $(a+b)(c+d)$.
2. $(m-n)(x+y)$.
3. $(x^2+y^2)(2a-3b)$.
4. $(3r+5s)(2a-7b)$.
5. $(x^2+y^2)(m^2-n^2)$.

VII

1. $(a+b+c)(a+b-c)$.
2. $(x+y-5)(x+y+5)$.
3. $(m-n-2a)(m-n+2a)$.
4. $(3x-5y+2z)(3x-5y-2z)$.
5. $(x^2+y^2-xy)(x^2+y^2+xy)$.

VIII

1. $(a+b-c)(a-b+c)$.
2. $(x+y-z)(x-y+z)$.
3. $(2m-3n+4a)(2m+3n-4a)$.
4. $(3a-2b+5c)(3a+2b-5c)$.
5. $(x^2+y^2+z^2)(x^2-y^2-z^2)$.

137. Factoring. The following types of factoring have all been studied. (See §§ 56–61.)

Exercise 123

Reduce each expression to prime factors:

I. $21a^3b^2c^2-28a^4b^4c+35a^5b^4=7a^3b^2(3c^2-4ab^2c+5a^2b^2)$.

(A polynomial with a monomial factor.)

1. $12a^2b^3-16a^3b^2+28a^3b^3$.
2. $7m^2n^2-14m^3n+21mn^3$.
3. $17x^5y^5-34x^4y^4+51x^3y^3$.
4. $26a^7b^5-39a^5b^7+13a^5b^5$.
5. $11a^4b^5c^6+22a^5b^6c^7-33a^6b^7c^8+44a^7b^8c^9$.

II. $9m^2-12mn+4n^2=(3m-2n)(3m-2n)$.

(A trinomial that is the square of a binomial.)

1. $n^2-8n+16$.
2. x^2+6x+9.
3. $16a^2-24ab+9b^2$.
4. $25m^2+40mn+16n^2$.
5. $x^4+2x^2y^2+y^4$.
6. $9x^6+12x^3y^3+4y^6$.

III. $m^2n^2-4=(mn-2)\ (mn+2)$.

(A binomial that is the difference of two squares.)

1. x^2-y^2. 2. $25a^2-36$. 3. $36m^2n^4-49x^4y^6$.
4. $(x-y)^2-z^2$. 5. $(2a+3b)^2-25$.

IV. $a^2-8ab+15b^2=(a-3b)\ (a-5b)$.

(A trinomial that is the product of two binomials with a common term.)

1. $n^2-10n+21$.
2. x^2-x-20.
3. $m^2-11mn+24n^2$.
4. $a^2+9ab+20b^2$.
5. $r^2-3r-40$.
6. $k^2+7k-60$.

V. $3x^2+5x+2=(x+1)\ (3x+2)$.

(A trinomial that is the product of two binomials with like terms.)

1. $6a^2-13a+6$.
2. $6x^2+5xy-6y^2$.
3. $7n^2-17n-12$.
4. $4r^2+8rs+3s^2$.
5. $12m^2n^2-25mn+12$.
6. $6k^6-37k^3+6$.

VI. $ax-nx-ay+ny=x(a-n)-y(a-n)=(x-y)(a-n)$.

(A polynomial that is the product of two polynomials with unlike terms.)

1. $2ax+3am-2bx-3bm$.
2. $a^2m^2+b^2m^2-a^2n-b^2n$.
3. $a^5-a^3x+a^2x^2-x^3$.
4. $6ax-4ay+9bx-6by$.
5. $ax-ay+bx-by+cx-cy$.

VII. $x^4-5x^2y^2+4y^4=(x^2-4y^2)\ (x^2-y^2)=$
$(x-2y)\ (x+2y)\ (x-y)\ (x+y)$.

(An expression that requires more than one process to reduce it to prime factors.) (See § 62.)

1. $7x^2-21xy+14y^2$.
2. a^3-9a^2+18a.
3. $m^4-2m^2n^2+n^4$.
4. a^4-b^4.
5. $81x^4-16$.
6. a^5-13a^3+36a.

7. $3x^4-x^2y^2-2y^4$.
8. $a^2m^2-a^2n^2-b^2m^2+b^2n^2$.
9. $a^3-5a^2-4a+20$.
10. $16m^4-8m^2n^2+n^4$.
11. $4x^2-16x+16$.
12. x^8-y^8.
13. $(a-b)^4-2(a-b)^2(x+y)^2+(x+y)^4$.
14. $32a^4-16a^2b^2+2b^4$.

Exercise 124. Miscellaneous Expressions

State under which of the preceding types each of the following may be factored, and find its factors:

1. a^2-4y^2.
2. a^4-4a^2.
3. a^3b-ab^3.
4. $8x^3-18xy^2$.
5. $64-n^4$.
6. a^2-2a-8.
7. $6x^2-13xy+6y^2$.
8. $4a^2-15a-4$.
9. $3a^2-a-2$.
10. $3x^3-6x^2y+3xy^2$.
11. $a^2+2ab+b^2-9$.
12. $(m+n)^2-(a-b)^2$.
13. $(x-y)^2-(x-y)$.
14. $a^5-4a^4+4a^3$.
15. $5n^2-25n+30$.
16. $a^2-b^2-2bc-c^2$.
17. x^2-y^2+x-y. (Divide by $x-y$.)
18. x^3y-xy^3+x-y.
19. $a^2-b^2+2bc-c^2$.
20. $25-x^2+2xy-y^2$.
21. $16-(a-b)^2$.
22. $4a^2+11ay+6y^2$.
23. a^4-a^2-20.
24. $48a^4-3b^4$.
25. a^4-5a^2+4.
26. $a+b+a^2-b^2$.
27. $ax+bx+ay+by$.
28. $a^2+2ab+b^2-81$.
29. $9a^2-21a+10$.
30. $100-29a^2+a^4$.
31. $a^2b^2-2abcd+c^2d^2$.
32. $8m^2-11mn+3n^2$.
33. a^2+a-b^2+b.
34. $3x^2yz-6xy^2z^2+3y^3z^3$.
35. $9x^4-15x^2-36$.
36. $(a-b)^2-2(a-b)c+c^2$.
37. $(x+y)^2-(x+y)\ (a-b)-20(a-b)^2$.

138. Equations. State the fundamental axioms of the equation. (See § 19.)

Exercise 125

Solve the following equations and check:

1. $3x+8-5x-7=3+2x-22$.
2. $2(5n+1)+3(7-n)=-12$.

3. $8x-5(5x+3)+3=3(7-2x)$.

4. $7(2n-3)-6(2n+3)=8(3n-4)-3(7-2n)$.

Solve the following literal equations for x and check:

5. $3x(x+a)-(x+a)^2=2x^2$.

6. $(x+m)^2-(x-m)^2=16m^2$.

7. $a(x-a)-b(x-b)=(a-b)^2$.

Exercise 126. Quadratic Equations

Solve each of the following by factoring and check:

1. $x^2-x-6=0$.
2. $x^2-x=0$.
3. $a^3-5a^2+4a=0$.
4. $x^2-x=12$.
5. $9x^2-17x=2$.
6. $12a^2-25a=-12$.

Solve each of the following by completing the square or by the formula and check: (See §§ 124 and 125.)

7. $2x^2-5x+2=0$.
8. $4x^2-3x-1=0$.
9. $6x^2-13x+2=0$.
10. $3x^2-11x=4$.
11. $2(a-1)^2+2a^2=5a(a-1)$.
12. $(2n+1)^2-(n+1)^2=(3n-2)^2$.

139. Highest common factor and lowest common multiple. *The highest common factor of two or more expressions is the product of the factors that are common to all the expressions.*

The lowest common multiple of two or more expressions is the product of all the different factors of the expressions, each factor used as many times as it occurs in any one of the expressions.

Exercise 127

Find the H. C. F. of the following:

1. $12a^3b^2$, $16a^2b^3$, and $20a^2b^2$.
2. $25m^2n^3$, $100m^3n^2$, and $75m^3n^3$.
3. a^2-ab, $a^3-2a^2b+ab^2$, and $2a^3-3a^2b+ab^2$.
4. x^2-x-6, x^2-5x+6, $x^2-7x+12$, and $ax-3a$.
5. x^4-y^4, x^2-y^2, $x-y$, and $x^2-2xy+y^2$.
6. $3ab-3b$, $2an-2n$, and a^2-1.
7. x^2+6x+5, $x^2+3x-10$, and $x^2+2x-15$.

Find the L. C. M. of the following:

8. 36, 16, 10, and 35.

9. $6a^2b$, $12a^3b$, and $10a^3b^3$.

10. $ax+ay$, x^2-y^2, and $x^2+2xy+y^2$.

11. x^2-x-6, x^2+5x+6, and x^2-9.

12. $3a^2+5ab+2b^2$, $3a^2-5ab+2b^2$, and a^2-b^2.

13. n^2-3n+2, n^2-4n+3, and n^2-5n+6.

140. Fractions.

State what fundamental operations may be performed on the terms of a fraction without changing its value. (See § 68.)

Exercise 128

Reduce the following fractions to lower terms:

1. $\dfrac{6axy}{9xyz}$. **2.** $\dfrac{-28a^3b^4c^7}{42a^4b^3c^6}$. **3.** $\dfrac{-10x^7y^8z^9}{-40x^5y^8z^{11}}$.

4. $\dfrac{x^2-y^2}{ax-ay}$. **5.** $\dfrac{a^2-4b^2}{a^2-3ab+2b^2}$. **6.** $\dfrac{9x^3y^3-9x^2y^4}{12x^4y^2-12x^3y^3}$.

7. $\dfrac{n-m}{m^2-n^2}$. **8.** $\dfrac{a^2-b^2}{b^2-2ab+a^2}$. **9.** $\dfrac{3-x}{x^2-5x+6}$.

Exercise 129

Supply the missing terms in the following:

1. $\dfrac{2x}{3y}=\dfrac{\quad}{12x^2y^2}$. **2.** $\dfrac{a}{a-b}=\dfrac{\quad}{a^2-b^2}$

3. $\dfrac{5a}{1}=\dfrac{\quad}{a-2b}$. **4.** $\dfrac{a}{b-a}=\dfrac{\quad}{a-b}$.

5. $\dfrac{1-n}{(1-2n)\ (1+2n)}=\dfrac{\quad}{(2n-1)\ (2n+1)}$.

6. $\dfrac{x+3}{x-2}=\dfrac{\quad}{x^2-5x+6}$. **7.** $\dfrac{4ax}{-5by}=\dfrac{-12abxy}{\quad}$.

8. $\dfrac{m-n}{m+n}=\dfrac{\quad}{3am^2-3an^2}$. **9.** $\dfrac{x-z}{x-y}=\dfrac{\quad}{y^2-x^2}$.

Exercise 130

Change each of the following fractions to mixed expressions:

1. $\dfrac{12a^2-20a+3}{4a}=3a-5+\dfrac{3}{4a}$ Ans.
2. $\dfrac{3x^3-9x^2-12x-2}{3x}$.
3. $\dfrac{9a^2+b^2}{3a-b}$.
4. $\dfrac{8x^3+27}{2x-3}$.

Change each of the following mixed expressions to a fraction:

5. $x+3+\dfrac{3}{2x}=\dfrac{2x^2+6x+3}{2x}$ Ans.
6. $a+5+\dfrac{2a-3}{2a}$.
7. $a^2-ab+b^2-\dfrac{2b^3}{a+b}$.
8. $x-3-\dfrac{5x-9}{x+3}$.
9. $\dfrac{a+b}{a-b}+1$.
10. $\dfrac{x+y}{x-y}-1$.

Exercise 131

Perform at sight the following additions and subtractions:

1. $\dfrac{x}{y}-1$.
2. $\dfrac{a}{b}-\dfrac{b}{a}$.
3. $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$.
4. $\dfrac{3}{x-y}+\dfrac{2}{x+y}$.
5. $\dfrac{x}{2}-\dfrac{2x}{3}+\dfrac{3x}{4}$.
6. $\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}$.
7. $\dfrac{m+n}{n}+\dfrac{n}{m-n}$.
8. $\dfrac{3}{x^2-9}-\dfrac{2}{x^2-5x+6}$.
9. $\dfrac{x}{y}-\dfrac{x}{x+y}-\dfrac{x^2}{y(x+y)}$.

Perform the following additions and subtractions:

10. $\dfrac{2}{a-1}-\dfrac{3}{1-a}=\dfrac{2}{a-1}+\dfrac{3}{a-1}=\dfrac{5}{a-1}$ Ans.
11. $\dfrac{3}{x-2}-\dfrac{2}{x+2}-\dfrac{2}{2-x}$.
12. $\dfrac{3}{x-y}+\dfrac{2}{x+y}-\dfrac{3y}{y^2-x^2}$.

Exercise 132

Perform the indicated operations in each of the following and reduce the result to its simplest form:

1. $\dfrac{2}{3}\cdot\dfrac{4}{5}\cdot\dfrac{7}{8}$.
2. $\dfrac{3}{4}\cdot\dfrac{5}{6}\div\dfrac{5}{8}$.
3. $\dfrac{6x^2y^3}{35a^3b^2}\cdot\dfrac{14a^2b^3}{27x^4y^4}$.

4. $\dfrac{36a^3b^2}{25x^2y}\cdot\dfrac{16xy}{27ab^3}\div\dfrac{8ax}{9by}$. 5. $\dfrac{n^2-n-6}{n^2-6n+9}\cdot\dfrac{n^2+n-6}{n^2-4}$.

6. $\dfrac{a-b}{a-c}\cdot\dfrac{b-c}{b-a}\div\dfrac{b-a}{a-c}$. 7. $\dfrac{x^2y^2-1}{x^2y^2-9}\cdot\dfrac{x^2y^2-xy-6}{x^2y^2-xy-2}$.

8. $\dfrac{x-y}{(a-b)\ (b-c)}\cdot\dfrac{x+y}{(a-c)\ (b-a)}\div\dfrac{x^2+y^2}{(c-a)\ (c-b)}$.

9. $\dfrac{a^2-2ab+b^2-c^2}{a^2+2ab+b^2-c^2}\cdot\dfrac{a^2-b^2-2bc-c^2}{a^2-b^2+2bc-c^2}$.

10. $\dfrac{5a^2-\ 7ab+2b^2}{12a^2-25ab+12b^2}\div\dfrac{10a^2+11ab-6b^2}{8a^2-\ 2ab-3b^2}$.

141. Fractional equations.

Exercise 133

Solve the following equations and check:

1. $\dfrac{x}{2}+\dfrac{x}{3}=10$. 2. $\dfrac{x}{2}-\dfrac{x}{5}=6$. 3. $\dfrac{y}{2}+\dfrac{y}{3}+\dfrac{y}{4}=-26$.

4. $\dfrac{n+3}{4}+\dfrac{n+1}{3}-\dfrac{n+8}{5}=6$. 5. $\dfrac{2m+1}{3}+\dfrac{m-4}{2}=\dfrac{5m+1}{3}$.

6. $\dfrac{n-2}{(n-1)\ (n+1)}-\dfrac{1}{n-1}=\dfrac{3}{n+1}$.

7. $\dfrac{x-a}{3}+\dfrac{x+a}{5}=x-2a$. (Solve for x.)

8. $\dfrac{x}{a+b}-\dfrac{x}{a-b}=\dfrac{1}{a+b}$. (Solve for x.)

9. $\dfrac{1}{x-2a}+\dfrac{1}{x+a}=\dfrac{m+n}{x^2-ax-2a^2}$. (Solve for x.)

142. Simultaneous systems of equations.

Exercise 134

Solve the following simultaneous linear systems and check:

(Follow the plan of eliminating one of the unknowns by addition or subtraction. Method I, § 87.)

1. $x+y=1$
$x-y=5$.

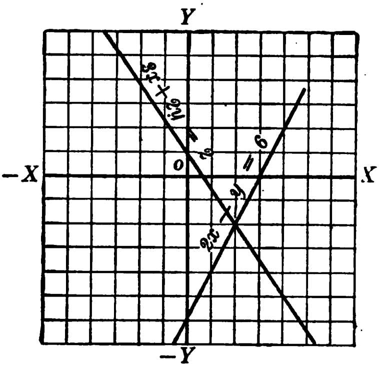
2. $2x-y=6$
$3x+2y=2$.

3. $2a+3b=1$
$3a-b=18$.

4. $5x+4y=8$
$7x-3y=-6.$

5. $3m-5n=4$
$5m+2n=-14.$

On the accompanying figure are the graphs of the equations in No. 2.



6. Solve No. 4 by graphing.

Solve the following for x and y:

7. $ax+by=2ab$
$ax-by=0.$

8. $mx+ny=k$
$ax+by=c.$

Exercise 135

Solve the following systems of equations, one linear and one quadratic, and check:

(Hint. Solve the linear equation for one unknown in terms of the other and substitute this value for that unknown in the quadratic.)

1. $x+y=5$
$x^2+y^2=25.$

2. $x-y=3$
$xy=10.$

3. $x^2-y^2=16$
$x-y=2.$

4. $x^2+y=12$
$x+y=0.$

5. $2x^2=y(x+6)$
$x+2y=7.$

6. $x^2+y^2-4x=21$
$x-y=1.$

Exercise 136. Problems

1. If the sum of two numbers is 15, and one is x, what is the other?

2. The sum of two numbers is 13. One-third the larger is one more than one-half the smaller. Find the numbers.

3. Write four consecutive numbers beginning with x. Write three consecutive odd numbers of which $x+1$ is the first.

4. The sum of three consecutive numbers is 33. What are they?

5. Find four consecutive even numbers whose sum is 60.

6. Find three consecutive even numbers such that the sum of the first two exceeds the third by 28.

7. Find four consecutive numbers such that the product of the third and fourth exceeds the product of the first and second by 26.

8. Find three consecutive integers whose sum exceeds half the largest by 37.

9. The second of three numbers is 3 less than twice the first and the third is 3 more than three times the first. The sum of the numbers is 42. What are they?

10. Separate 47 into two parts such that one part shall exceed the other by 15.

11. If $2x+9$ represents 37, what number will $x+5$ represent?

12. James has $3.10 in dimes and quarters. If he has 3 more dimes than quarters, how many dimes has he?

13. The width of a rectangle is 3 inches more than half its length and its perimeter is 66 inches. Find its length and width.

14. The algebraic sum of three numbers is 7. The second is 11 less than the first and the third is 8 more than twice the second. Find the three numbers.

15. The average temperature of Thursday was 15° colder than that of Wednesday and the average temperature of Friday was 8° warmer than that of Thursday. Find the three temperatures if their sum was 11°.

16. John and James together have 40 cents. If twice the amount that John has be subtracted from 5 times the amount that James has the result is 25 cents. How much has each?

17. The first of the three angles of a triangle exceeds the second by 40° and the third angle equals half the sum of the other two. Find the angles.

(Hint. The sum of the angles of a triangle is 180°.)

18. A is twice as old as B and 20 years ago he was 4 times as old. Find their present ages.

19. The second of two numbers is 2 more than 3 times the first. If the second be subtracted from 5 times the first, the remainder is 12. Find the numbers.

20. Two-fifths of the sum of a certain number and 9 equals 20. What is the number?

21. Separate 48 into two parts such that one-half of one part added to one-third of the other part gives 22.

22. Separate 56 into two parts such that their quotient equals 4/3.

23. What number added to both the numerator and the denominator of 7/9 will make the resulting fraction equal to 9/10?

24. A sum of $600 was divided equally among a certain number of persons. If there had been 6 more persons, each would have received four-fifths as much. How many persons were there?

25. Find the time between 5 and 6 o'clock when the hands of the clock are together.

26. Find the time between 8 and 9 o'clock when the hands of a clock point in opposite directions.

27. Find the times between 5 and 6 o'clock when the hands of a clock are at right angles.

28. Find two numbers such that 3 times the first equals 5 times the second and the sum of 5 times the first and 2 times the second is 62.

29. I have $10.00 in dimes and quarters, 52 coins altogether. How many of each have I?

30. If 10 apples and 9 oranges cost 75 cents and at the same price 7 apples and 8 oranges cost 61 cents, find the cost of an apple and an orange.

31. There are three angles whose sum is 360°. The first is the supplement of one-third of the second and the first exceeds the third by 40°. Find the number of degrees in each angle.

32. A motor boat that can make 12 miles an hour in still water requires 6 hours to go 32 miles upstream and return. Find the rate of the stream.

33. Solve the formula $l=a+(n-1)d$ for n. Also for a and d.

34. Solve the formula $A=\dfrac{a(B+b)}{2}$ for b.

35. A man drove an automobile to a town 60 miles from his home and returned immediately by train. The whole trip required 4 hours and the average rate of the train was 16 miles per hour more than the average rate of the automobile. Find the two rates.

36. Find two numbers whose difference is 4 and the difference of whose squares is 64.

37. Find two numbers whose difference is 4 and the sum of whose squares is 40.

38. The hypotenuse of a right triangle is 2 inches longer than one side and 4 inches longer than the other. Find the three sides of the triangle.

Suggestion. Use the Pythagorean Proposition. *The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.*

39. A rectangle is 4 inches longer than it is wide and its area is 45 sq. in. Find its dimensions.

40. Find the dimensions of a rectangle whose area is 84 sq. in. and whose length is 2 inches less than twice its width.

41. Find the dimensions of a rectangular field whose area is 240 square rods and whose perimeter is 64 rods.

42. Find the dimensions of a rectangular field whose area is 30 acres and whose perimeter is one mile.

CHAPTER XII

ADDITIONAL FUNDAMENTAL PROCESSES

143. Laws on the order of operations.

The sum of several numbers is the same in whatever order they are taken, for $a+b+c=a+c+b=b+c+a=c+b+a$.

This is known as the **commutative law for addition.**

It is obviously true for the sum of positive and negative numbers, for $2-3+4=2+(-3)+4$. Since subtraction is considered as the addition of a negative number, the law holds for subtraction. That is, $2-3=-3+2$.

The same law for multiplication states that *the factors of a product may be taken in any order,* for $2\cdot3\cdot4=2\cdot4\cdot3=4\cdot3\cdot2$.

Since division may be considered as the multiplication by a reciprocal and $4\div2\cdot3=4\cdot\frac{1}{2}\cdot3=4\cdot3\cdot\frac{1}{2}=3\cdot4\cdot\frac{1}{2}$, the same law applies in division.

Now $4+(3+5)=(4+3)+5=5+(3+4)$, and $b+(a+c)=(a+b)+c$. Evidently *the sum of several numbers is the same in whatever order they are grouped.*

This is known as the **associative law for addition.**

Also $4(5\cdot3)=(4\cdot5)3=5(4\cdot3)$ and $a(b\cdot c)=(a\cdot b)c=b(a\cdot c)$. *The product of several factors is the same in whatever order the factors may be grouped* is the same law for multiplication.

In arithmetical problems involving additions, subtractions, multiplications, and divisions, it is necessary to perform the multiplications and divisions first, then the additions and subtractions may be performed in any convenient order.

Illustrative example.

$$3-4\div2+5\cdot2\cdot6\div12=3-\frac{4}{2}+\frac{5\cdot2\cdot6}{12}=3-2+5=6.$$

Exercise 137

Simplify each of the following:

1. $4 \cdot 7 - 12 \cdot 2 \div 3 - 20 \div 5 \cdot 6$.
2. $27 - 9 \cdot 9 \div 3 + 4 \cdot 0 \cdot 8 \div 16$.
3. $12 + 12 \div 3 - 7 \cdot 8 \div 14 - 2(7 - 4 \div 2)$.
4. $10 - 5 \cdot 20 \div 25 + 18 \div 9 \cdot 3 \div 6 - \frac{1}{2}(5 - 7 \cdot 3 \cdot 2 \div 6)$.
5. $(12 - 8 \div 4 \cdot 3)\ (6 + 5 \div 10 - 2 \cdot 3 \div 4) - 7 \cdot 4 \cdot 3 \div 14$.
6. $\frac{1}{2} \div \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} \div \frac{1}{2} - \frac{7}{2} \cdot \frac{3}{4} \div 6 \div \frac{7}{8} + 2 \cdot 5 \cdot 3$.

The symbols of algebra are so definite that there is seldom a question as to the order of operations in evaluating an algebraic expression.

Evaluate each of the following, if $a=2$, $b=3$, $x=1$, *and* $y=4$:

7. $a^2 - 2ab + b^2 - x^2 + 2xy - y^2$.
8. $x^2y^2 - 2abxy + a^2b^2 + x^2 - y^2 + a^2 - b^2$.
9. $a^3b^3 - 3a^2b^2xy + 3abx^2y^2 - x^3y^3$.

144. The signs of aggregation. The common signs of aggregation are the **parenthesis** ( ), the **bracket** [ ], the **brace** { }, and the **vinculum** $\overline{\qquad}$.

The vinculum has its most frequent use with the radical sign, $\sqrt{\quad}$. The bar of a fraction is a common sign of aggregation and is equivalent to a vinculum.

Whenever any one of these signs is used the expression enclosed is to be treated as a single quantity until the operation of removing the sign is performed. When simplifying expressions involving the use of more than one of the signs of aggregation, it is best to remove but one pair at a time beginning with the innermost.

Illustrative example.

$2a - \{-3[a - (a - \overline{b - c}) - 2b] + c\} =$

$2a - \{-3[a - (a - b + c) - 2b] + c\} =$

$2a - \{-3[a - a + b - c - 2b] + c\} = 2a - \{-3[-b - c] + c\} =$

$2a - \{+3b + 3c + c\} = 2a - \{3b + 4c\} = 2a - 3b - 4c$. Ans.

Exercise 138

Remove all signs of aggregation and simplify each of the following:

1. $18-2(5-3)+3(-2[-2-\overline{4-2}-3])-2(-4+5)$.
2. $x-[x-(-\{-x-y\}-2y)-3y]+2(3[x+1])$.
3. $2x-2[-2(-2\{-2x+\overline{2y-2z}\}-4z)+8y]-16z$.
4. $x-(y-z-[x-\{y-z-\overline{x-y-z}-x\}-y]-z)-y$.
5. $2x-[3y-(4z-5w-\overline{x+y+z+w}-3y)-x]$.
6. $m+(m-n)-[m-(m-n)-\overline{m+n}]-(m+n)$.
7. $2x-2[-2(3x-2\overline{x-y})]=2x-2[-2(3x-2x+2y)]=?$

Note. The vinculum must be observed closely; $-2\overline{x-y}$ is the same as $-2(x-y)$, but is not $-\overline{2x-y}$.

8. $(-\overline{2a-5b})-(-2a-5b)$. Ans. $10b$.
9. $x-(2x-[3x-\{4x-5\overline{x-1}-2\}-1]-1)$.
10. $a-2\overline{b-3c}-\{-a+\overline{2b+3c}-2[a-b-3\overline{a+b-c}]-4a\}$.

Make the following expressions the difference of two squares by using signs of aggregation:

11. $a^2-b^2-c^2+2bc=a^2-(b^2-2bc+c^2)$.
12. $x^2-y^2-2ax-z^2+2yz+a^2$.
13. $9-x^2-y^2-6a-2xy+a^2$.
14. $12a-9-4a^2+m^2-2mn+n^2$.
15. $25x^2-20bx+4b^2-60ay-36y^2-25a^2$.
16. $4x^4-4x^2-1+x^6-4x^5+4x$.

In the following collect all negative terms and express as a single negative quantity:

17. $x^4-3x^3+6x^2-2x+1=x^4+6x^2+1-(3x^3+2x)$.
18. $x^2-ax-bx+cb=x^2-(ax+bx)+cb=x^2-(a+b)x+cb$.
19. $x^2-ax-bx-cx+abc$.
20. $x^3-ax^2-bx^2-cx^2+abx+acx+bcx+abc$.

145. Detached coefficients. When required to find the product or the quotient of two polynomials that can be arranged with reference to the same letter, the work can be

shortened and the possibility of errors decreased by detaching coefficients. Note the following **illustrative examples:**

1. Multiply $3x^2-2x-7$ by $2x^2-3x-7$.

Old Process	Detaching coefficients	
$3x^2-2x-7$	$3-2-7$	$=-6$
$2x^2-3x-7$	$2-3-7$	$=-8$
$6x^4-4x^3-14x^2$	$6-4-14$	
$-9x^3+6x^2+21x$	$-9+6+21$	
$-21x^2+14x+49$	$-21+14+49$	
$6x^4-13x^3-29x^2+35x+49.$	$6-13-29+35+49$	$=48$

Supplying x gives $6x^4-13x^3-29x^2+35x+49$.

If 1 is substituted for x, notice the ease of checking for multiplicand $=-6$, multiplier $=-8$, and product $=48$.

Care must be taken to provide a 0 for each power of the literal number that is lacking in the arrangement.

2. Multiply $2x^4+3x^2-2x-4$ by x^3-3x^2+3 and check.

$$\begin{array}{rl}
2+0+3-2-4 & =-1 \\
\underline{1-3+0+3} & =+1 \\
2+0+3-2-4 & \\
-6+0-9+6+12 & \\
\underline{+6+0+9-6-12} & \\
2-6+3-5+2+21-6-12 & =-1
\end{array}$$

Supplying x gives $2x^7-6x^6+3x^5-5x^4+2x^3+21x^2-6x-12$.

3. Divide $15a^4-a+8a^2-1-19a^3$ by $5a^2-1-3a$.

Arranging and detaching coefficients gives

$$\begin{array}{r|l}
15-19+8-1-1 & 5-3-1 \\
\underline{15-9-3} & \overline{3-2+1} \\
-10+11-1 & 3a^2-2a+1.\text{ Ans.} \\
\underline{-10+6+2} & \\
+5-3-1 & \\
\underline{+5-3-1} &
\end{array}$$

On checking, the dividend $=2$, the divisor $=1$, and the quotient $=2$.

Exercise 139

Perform the indicated operation in each of the following, using detached coefficients:

1. Multiply a^3-3a^2+3a-1 by a^2-2a+1.
2. Multiply m^3-m^2+3m-5 by m^3+m^2+3m+5.
3. Divide $a^4-4a^3+6a^2-4a+1$ by a^2-2a+1.
4. Divide $m^4-3m^3-36m^2-71m-21$ by m^2-8m-3.
5. Multiply $1-7x^2+x^3+5x$ by $1+2x^2-4x$.

6. Multiply $a^4-4a^3b+6a^2b^2-4ab^3+b^4$ by $a^2-2ab+b^2$.
7. Divide a^4+a^2+1 by a^2+a+1. Also by a^2-a+1.
8. Multiply x^5-2x^3+3 by $1-x^2+x$.
9. Divide $m^6-6m^4+5m^2-1$ by m^3+2m^2-m-1.
10. Divide $m^4+4m^2n^2+16n^4$ by $m^2+2mn+4n^2$.
11. Divide m^5-n^5 by $m-n$. Also by $m+n$.
12. Multiply $m^4-m^3n+m^2n^2-mn^3+n^4$ by $m+n$.

146. Synthetic division. When the divisor is a binomial of the type $x-a$ the work can be still further abbreviated.

Illustrative examples.

1. Divide x^3-5x^2+7x-2 by $x-2$.

I. Detached coefficients.

$$\begin{array}{l|l}
1-5+7-2 & 1-2 \\ \cline{2-2}
\underline{1-2} & 1-3+1 \\
-3 & \\
\underline{-3+6} & \\
+1 & \\
+1-2 &
\end{array}$$

II.

$$\begin{array}{l|l}
1-5+7-2 & 1-2 \\ \cline{2-2}
\underline{-2} & 1-3+1 \\
-3 & \\
\underline{+6} & \\
+1 & \\
-2 &
\end{array}$$

Notice that II is obtained from I by omitting the first term of each partial product, for their subtraction is planned to eliminate the corresponding term of the dividend. Also notice that the first term of the dividend and of each partial dividend, 1, -3, $+1$, form the terms of the quotient in order and, since $+6$ is to be subtracted from $+7$, and -2 from -2, the form II may still further be compressed. See III and IV following. Notice that the quotient appears below.

III. or IV.

$$\begin{array}{l|l}
1-5+7-2 & 1-2 \\
\underline{-2+6-2} & \\
1-3+1 & \text{Ans. } x^2-3x+1.
\end{array}
\qquad
\begin{array}{l|l}
1-5+7-2 & +2 \\
\underline{+2-6+2} & \\
1-3+1 & \text{Ans. } x^2-3x+1.
\end{array}$$

Now -2 is the essential term of the divisor and multiplication by it changes every sign of the partial products. But we are to subtract each of these partial products, therefore we may change -2 to $+2$ and add the partial products as in IV.

2. Divide x^3-2x^2-5x+8 by $x-3$.

Detaching coefficients and following IV gives:

$$\begin{array}{l|l}
1-2-5+8 & +3 \\
\underline{+3+3-6} & \\
1+1-2. & 2
\end{array}
\qquad \text{Ans. } x^2+x-2+\frac{2}{x-3}.$$

Exercise 140

1. Divide $a^3-3a^2-4a+12$ by $a-3$. Also by $a-2$.
2. Divide x^3+3x^2+x-2 by $x+2$. Also by $x+1$.
3. Divide x^3+2x^2-6x+8 by $x+4$.
4. Divide $a^3-22a+15$ by $a+5$.
5. Divide $x^4+3x^3-5x^2+4x-4$ by $x-2$.
6. Divide $m^4-m^3+5m^2-9m-10$ by $m-2$.
7. Divide $8x^3-24x^2+36x-27$ by $2x-3$.

Suggestion. Divide first by $x-3/2$, then divide the quotient by 2.

8. Divide $16a^4-81$ by $2a-3$. Also by $2a+3$.
9. Divide $6(x-y)^2-7(x-y)-20$ by $3(x-y)+4$.
10. Divide $3x^3-13x^2+23x-21$ by $3x-7$.
11. Check Numbers 7, 8, 9, and 10 by detaching coefficients and dividing as in § 145.

147. Additional special products and factoring.

Type VIII. The square of a polynomial. (For Types I to VII see § 136.)

$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$

and $(a-b-c)^2=a^2+b^2+c^2-2ab-2ac+2bc$.

Stated as a **rule** this becomes—

The square of a polynomial is equal to the sum of the squares of the terms together with twice the product of each term and every one that follows it.

Exercise 141

Expand each of the following:

1. $(m+n+p)^2$.	2. $(x-y+z)^2$.
3. $(2x-3y-2z)^2$.	4. $(3m+2n+3a)^2$.
5. $(x+y-m-n)^2$.	6. $(x+w-3z+5)^2$.

The following are the squares of what polynomials?

7. $a^2+x^2+9y^2-2ax+6ay-6xy$.
8. $4m^2+25n^2+36p^2+20mn-24mp-60np$.
9. $a^2+4b^2+9c^2+49-4ab-6ac-14a+12bc+28b+42c$.

Type IX. The cube of a binomial.

Expand $(a+b)^3$. Expand $(a-b)^3$.

Exercise 142

Translate each of the formulas obtained by the expansions of $(a+b)^3$ and $(a-b)^3$ into a rule as in VIII and apply in the expansion of each of the following:

1. $(x+y)^3$. 2. $(2-a)^3$. 3. $(2x+y)^3$.
4. $(3m-2n)^3$. 5. $(2ab+c)^3$. 6. $(x^2-3y^2)^3$.

What are the factors of the following?

7. $27-27x+9x^2-x^3$.
8. $64-240a+300a^2-125a^3$.
9. $8a^3-36a^2b+54ab^2-27b^3$.
10. $a^6-9a^4x+27a^2x^2-27x^3$.

Type X. The factors of the sum or the difference of two cubes. Find by multiplication the products

$(a+b)\ (a^2-ab+b^2)$ and $(a-b)\ (a^2+ab+b^2)$.

Translate into a rule reading:—"*The factors of the sum of two cubes are the sum of the cube roots . . . ,*" and "*the factors of the difference of two cubes are . . . .*"

Exercise 143

Write at sight the following products:

1. $(x+y)\ (x^2-xy+y^2)$. 2. $(x-y)\ (x^2+xy+y^2)$.
3. $(m+2)\ (m^2-2m+4)$. 4. $(2a-b)\ (4a^2+2ab+b^2)$.
5. $(x^2+y^2)\ (x^4-x^2y^2+y^4)$.
6. $(2a^2-3b)\ (4a^4+6a^2b+9b^2)$.

Write at sight the factors for each of the following:

7. m^3+n^3. 8. m^3-n^3. 9. x^3+1.
10. x^3-1. 11. a^3+8. 12. a^3+8b^3.
13. $27m^3-1$. 14. $8n^3+27r^3$. 15. $m^3n^3+y^3$.
16. $125+y^3$. 17. $a^3b^3-x^3y^3$. 18. c^3+343.
19. a^6+1. Treat as $(a^2)^3+1$. 20. $125-m^6$.
21. $8m^6-27n^6$. 22. $64a^9-x^3$. 23. $x^{12}+1$.
24. x^6-64. 25. x^9-y^6. 26. $(x-y)^3-8$.

Type XI. Find the product of a^2+ab+b^2 and a^2-ab+b^2 in two ways:

(1) by detaching coefficients,

(2) by treating the trinomials as the sum and the difference of two terms, i.e.,

$$[(a^2+b^2)+ab]\,[(a^2+b^2)-ab].$$

Exercise 144

Write the product for each of the following:

1. $(x^2+xy+y^2)\,(x^2-xy+y^2)$.
2. $(n^2+3mn+9m^2)\,(n^2-3mn+9m^2)$.
3. $(4a^2-6ab+9b^2)\,(4a^2+6ab+9b^2)$.
4. $(x^4-x^2y^2+y^4)\,(x^4+x^2y^2+y^4)$
5. $(16a^2-4a+1)\,(16a^2+4a+1)$.

Study the products obtained in Numbers 1–5, then reverse the process and factor each of the following, using the plan of Number 6:

6. $a^4+a^2b^2+b^4=a^4+2a^2b^2+b^4-a^2b^2$
$=(a^2+b^2)^2-a^2b^2=[a^2+b^2+ab]\,[a^2+b^2-ab]$.

7. x^4+x^2+1. 8. $1+a^2+a^4$. 9. $x^8+x^4y^4+y^8$.

10. $a^4+3a^2+4=a^4+4a^2+4-a^2=?$

11. a^4-3a^2+9. 12. $a^4+a^2b^2+25b^4$.

13. $x^4-7x^2+1=x^4+2x^2+1-9x^2=?$

14. $4a^4+7a^2b^2+4b^4$.

15. $x^4+4=x^4+4x^2+4-4x^2=?$

16. $4x^4+1$. 17. $64+x^4$. 18. $64x^4+1$.

19. $x^4+\frac{1}{4}$. 20. x^4-3x^2+1.

21. x^4-11x^2+1. 22. x^4-18x^2+1.

23. Make and factor three others of the type of 20, 21, and 22.

24. $x^4-14x^2+1=x^4+2x^2+1-16x^2=?$

25. x^4-23x^2+1. 26. x^4-34x^2+1.

27. Make and factor three others of the type of 24, 25, and 26.

148. The Remainder Theorem and the Factor Theorem.

A **rational number** is one that can be expressed as a single whole number or as the quotient of two whole numbers.

If any expression of the form x^3+2x^2+3x+2 is divided by a binomial of the form $x-2$, the remainder is the same as if $+2$ were substituted for x.

Show by synthetic division that $(x^3+2x^2+3x+2)\div(x-2)$ gives $x^2+4x+11$ and remainder $+24$. Also substituting 2 for x, it gives $(2)^3+2(2)^2+3(2)+2=24$.

In general, $(x^2+ax+b)\div(x-n)$ gives $x+a+n$ and remainder n^2+an+b (check by division), therefore we have the **Remainder Theorem:**

If any rational integral expression in x is divided by $x-n$, the remainder is the same as if n were substituted for x.

Evidently, when the remainder becomes 0, $x-n$ is a factor and we have the **Factor Theorem:**

If any rational integral expression in x becomes 0 when n is substituted for x, then $x-n$ is a factor of the expression.

This theorem is useful in factoring expressions of a degree higher than the second.

Illustrative examples.

1. Factor x^3+3x^2+3x+2. Evidently, if $x-n$ is a factor of this expression, then n must be a factor of 2. Now the factors of 2 are $+1$, -1, $+2$, and -2. If we substitute $+1$ for x, the expression becomes $1+3+3+2$, or 9. If we substitute -1, it becomes $-1+3-3+2$, or $+1$. Similarly $+2$ gives $+28$, but -2 gives 0, therefore $x-(-2)$ or $x+2$ is a factor. By synthetic division, the quotient is x^2+x+1 and the remainder is 0.

2. Factor x^3-2x^2-5x+6. The factors of 6 are $+1$, -1, $+2$, -2, $+3$, -3, $+6$, and -6. Substituting each of these in turn for x, we find that $+1$, -2, and $+3$ each makes the expression 0. Therefore $x-1$, $x+2$, and $x-3$ are the required factors. (Check by multiplication.)

3. Factor $x^3-ax^2-14a^2x+24a^3$. Some of the factors of $24a^3$ are $+a$, $-a$, $+2a$, $-2a$, $+3a$, $-3a$, $+4a$, $-4a$, ... Of these $+2a$, $+3a$, and $-4a$ makes the expression 0. Therefore $x-2a$, $x-3a$, and $x+4a$ are the required factors.

Exercise 145

Factor:

1. x^3-7x^2+6.
2. x^3-7x-6.
3. a^3-10a^2+9.
4. $a^3-20a+32$.
5. x^3-4x^2+5x-2.
6. x^3-6x^2+5x+6.
7. x^3+3x^2-6x-8.
8. $a^3+12a^2+12a-45$.
9. m^4-3m^3+5m-2.
10. a^3-5a^2+4a+4.
11. $a^4-3a^3-a^2+a+6$.
12. $m^5-3m^2+24m-68$.
13. $m^3-3m^2n+4mn^2-4n^3$. (Try $m=+n$, $-n$, $+2n$, etc.)
14. $a^3-2a^2n-5an^2+6n^3$.
15. $a^3-5a^2y+18y^3$.

149. Typeform $a^n \pm b^n$. (The sum or difference of two like powers.)

We have learned how to factor a number of expressions that come under this typeform such as x^2-y^2, x^3-y^3, x^3+y^3 . . . and have discovered that $x+y$ and $x-y$ are included among the factors.

The factor theorem makes clear the rule for determining when either or both are divisors. If $+b$ is substituted for a in a^n-b^n, the expression becomes $(+b)^n-b^n$ which equals 0. Therefore $a-b$ is a factor of a^n-b^n for any integral value of n.

If $-b$ is substituted for a, a^n-b^n becomes $(-b)^n-b^n$, or $+b^n-b^n$ when n is even and it becomes $-b^n-b^n$, or $-2b^n$, when n is odd. Therefore $a+b$ is a factor of a^n-b^n when n is even but not when n is odd.

Now if $+b$ is substituted for a in a^n+b^n, the expression becomes b^n+b^n, or $2b^n$, no matter whether n is odd or even. Therefore $a-b$ is never a divisor of a^n+b^n. But if $-b$ is substituted for a, a^n+b^n becomes $(-b)^n+b^n$ which equals 0 when n is odd and $2b^n$ when n is even. Therefore $a+b$ is a divisor of a^n+b^n when n is odd. This gives the **rule:**

The expression a^n-b^n with n an integer is always divisible by $a-b$. It is divisible by $a+b$ when n is even.

The expression a^n+b^n is never divisible by $a-b$. It is divisible by $a+b$ when n is odd.

Exercise 146

1. Divide x^5-y^5 by $x-y$ using synthetic division.

Solution.

$$\begin{array}{l|l} 1+0+0+0+0-1 & \underline{+1} \\ \underline{\quad 1+1+1+1+1} & \\ 1+1+1+1+1 & \end{array}$$

Ans. $x^4+x^3y+x^2y^2+xy^3+y^4$.

2. Divide x^5+y^5 by $x+y$ following the plan of Ex. 1.
3. Write the quotient of $(x^7-y^7)\div(x-y)$.
4. Write the quotient of $(x^7+y^7)\div(x+y)$.

Factor each of the following:

5. x^5-32. (Treat as x^5-2^5.)
6. x^5+243.
7. a^6+b^6. [Treat as $(a^2)^3+(b^2)^3$.]
8. a^9+b^9.
9. a^6+b^9. [Treat as $(a^2)^3+(b^3)^3$.]
10. a^6+125.

Exercise 147. Miscellaneous Exercises

Factor each of the following:

1. x^6-64.
2. x^6+64.
3. x^9+64.
4. $x^{12}+y^{12}$.
5. $x^{12}-y^{12}$.
6. $a^4+2a^3y-2ay^3-y^4$.
7. $x^2y^2+17xy+16$.
8. y^2-x^2-2x-1.
9. $x^2y^2+25-9z^2-10xy$.
10. x^8-34x^4+1.
11. x^2-y^2+x+y.
12. x^3-x^2+3x+5.
13. x^4+2x^2+9.
14. x^4-3x^2+9.
15. x^6+125y^3.
16. x^3-15x^2+250.
17. $10x^4-47x^2+42$.
18. a^6-64b^3.
19. $75a^2b^2-108c^2d^2$.
20. x^3-y^3+x-y.
21. x^3+x^2+x+1.
22. x^8+4.
23. $2x^2+3x-2$.
24. a^3-a^2-5a+2.
25. $36r^4-21r^2+1$.
26. $125m^3-150m^2+45m-2$.
27. $10x^2+3x-18$.
28. $abx^3+x+ab+1$.
29. $m^2-n^2+m^3-n^3$.
30. $a^2(a^2-1)-b^2(b^2-1)$.
31. $aby^3+y+ab+1$.
32. $7a^3x^2+49a^2x+84a$.
33. $(m-n)^2-9(m-n)-36$.
34. a^7+b^7.
35. $2a^3+7a^2+4a-4$.
36. y^4+y^3-3-3y.
37. $3x^6+8x^4-8x^2-3$.
38. $(a^3+1)^3-(b^3-1)^3$.
39. x^9-y^9.

150. Fractions.

Since a fraction is an indicated division it has three signs: (1) the sign of its numerator, or the dividend, (2) the sign of its denominator, or the divisor, (3) the sign of the value of the fraction, or the quotient. This is placed before the fraction and on a line with its bar. (See § 69.)

Any two of the signs of a fraction may be changed without changing the value of the fraction. The following are equivalent fractions: $\frac{a}{b}, \frac{-a}{-b}, -\frac{-a}{b}$, and $-\frac{a}{-b}$.
If $a=10$ and $b=2$ the value of each fraction is $+5$.

Similarly $\frac{a}{b-c}=\frac{-a}{c-b}=-\frac{a}{c-b}=-\frac{-a}{b-c}$.
Check by letting $a=6$, $b=4$, and $c=2$.

It will be recalled that the sign of the product of several positive and negative factors is $+$ if there is an even number of negative factors and $-$ if there is an odd number of negative factors. (See § 39.) This leads to the following:

Rule. *If the signs of an even number of factors are changed, the sign of their product is not changed. If the signs of an odd number of factors are changed, the sign of their product is changed.*

Exercise 148

1. Show that $(x-1)(x-2)=(2-x)(1-x)=-(x-1)(2-x)$ $=-(1-x)(x-2)$ by finding each of the four products by multiplication.

2. Write the product $(x-1)(x-2)(x-3)$ in as many different ways as possible.

Hint. $(x-1)(2-x)(3-x)$.
Check by letting $x=5$ in each.

3. Write the fraction $\frac{x}{2b-a}$ in several equivalent forms. Check by letting $a=5$, $b=4$, and $x=6$.

4. Write the fraction $\frac{a-b}{c-d}$ in three other equivalent forms and check with $a=5$, $b=3$, $c=2$, and $d=1$.

5. Write the fraction $\frac{b-c}{(a-b)\ (a-c)}$ in three other equivalent forms and without changing the sign of the value of the fraction check in each.

6. The fraction $\frac{a-b}{(a-c)\ (b-c)\ (a-d)}$ may be written in seven other equivalent forms without changing the sign of the value of the fraction. Write these forms.

151. An important assumption of fractions is the following:

Axiom. *The value of a fraction is not changed if both numerator and denominator are multiplied or divided by the same quantity.* (See Axiom VI, § 68.)

Under this axiom we have simplified fractions, or reduced them to their lowest terms, by removing all common factors from the numerator and denominator.

Exercise 149

Reduce each of the following fractions to its lowest terms:

1. $\frac{x-y}{x^3-y^3}$.
2. $\frac{x^2-4}{x^3-8}$.
3. $\frac{a^2-a-2}{a^3-3a^2+3a-2}$.
4. $\frac{x^3-y^3}{x^5-y^5}$.
5. $\frac{8x^3+1}{4x^4-17x^2+4}$.
6. $\frac{x^4-y^4}{x^6+y^6}$.
7. $\frac{x^3-3x^2+4}{x^4-10x+4}$.
8. $\frac{b^2-2bc+c^2-d^2}{d^2-2cd+c^2-b^2}$.
9. $\frac{x^6+y^6}{x^{10}+y^{10}}$.
10. $\frac{a^2-4b^2+8bc-4c^2}{a^3+8(b-c)^3}$.

152. In arithmetic, what is a proper fraction? Give an illustration. What is an improper fraction? Give an illustration.

An **improper algebraic fraction** is one the numerator of which contains a power of a literal number equal to or higher than the highest power of the same number in the denominator.

How would you define a proper algebraic fraction?

Exercise 150

Give the rule for changing a mixed expression to an equivalent improper fraction. Change each of the following mixed expressions into its equivalent improper fraction:

1. $x-y+\dfrac{x^3+y^3}{x^2+xy+y^2}$.
3. $x^2-xy+y^2-\dfrac{x^4+y^4}{x^2+xy+y^2}$.
2. $x^2-x-2+\dfrac{2}{x^2+x-2}$.
4. $x^4-x^3+x^2-x+1+\dfrac{2}{x+1}$.

Give the rule for changing an improper fraction into its equivalent mixed expression. Change each of the following fractions into mixed expressions:

5. $\dfrac{a^5-b^5}{a+b}$.
6. $\dfrac{x^3-3x^2-5}{x^2-x-1}$.
7. $\dfrac{x^5}{x^2+x+1}$.
8. $\dfrac{a^4+a^2b^2+b^4+3}{a^2+ab+b^2}$.
9. $\dfrac{a^5+c^5}{a-c}$.

153. Addition and subtraction of fractions.

Give the rule for finding the algebraic sum of several fractions with different denominators.

Exercise 151

Combine the following:

1. $\dfrac{x+1}{x^2-5x+6}+\dfrac{x+2}{x^2-7x+12}+\dfrac{x+3}{x^2-6x+8}$.
2. $\dfrac{1}{(a-b)\,(b-c)}-\dfrac{2}{(b-a)\,(b-c)}$. Ans. $\dfrac{3}{(a-b)\,(b-c)}$.
3. $\dfrac{2}{x+1}-\dfrac{3}{x-1}+\dfrac{1}{1-x^2}$.
4. $\dfrac{a+1}{(a-4)\,(c-2)}+\dfrac{a+4}{(a-1)\,(2-c)}$.

5. $\frac{m}{(m-n)(m-p)}+\frac{n}{(n-p)(n-m)}+\frac{p}{(p-m)(p-n)}.$

6. $\frac{x-3a}{x^2-3ax+9a^2}+\frac{3ax-2x^2}{x^3+27a^3}+\frac{1}{3a+x}.$

7. $\frac{a-1}{a^2-a+1}+\frac{2}{a^4+a^2+1}+\frac{a+1}{a^2+a+1}.$

8. $\frac{2}{a+3}-\left\{3-\frac{1}{a-3}-\left[\frac{1}{a^2-9}+\frac{a}{a-3}\right]\right\}.$

9. $\frac{(x-y)(z-y)}{y(y-a)}+\frac{xz}{ay}+\frac{(x-a)(z-a)}{a(a-y)}.$

154. Multiplication and division of fractions.

Exercise 152

Perform the indicated operations in each of the following and reduce to lowest terms:

1. $\frac{a^2-b^2}{b^3-a^3}\cdot\frac{a^4+a^2b^2+b^4}{b^3+a^3}.$ Ans. -1.

2. $\frac{ab}{a+b}\cdot\left(\frac{a}{b}-\frac{b}{a}\right)\cdot\frac{1}{a-b}.$

3. $\left(\frac{xy}{x+y}\right)\left(\frac{x+y}{x}+\frac{y}{x+y}\right).$

4. $\left(a+2b+\frac{b^2}{a}\right)\left(a-2b+\frac{b^2}{a}\right)\cdot\frac{a^2}{a^2-b^2}.$

5. $\left(x-\frac{y^2}{x}\right)\div\left(\frac{y^2}{x}+2y+x\right)\cdot\frac{x+y}{x-y}.$ Ans. 1.

6. $\left(a-b-\frac{a^2+b^2}{a+b}\right)\left(a+b-\frac{a^2+b^2}{a-b}\right).$

7. $\left(\frac{x}{a+x}-\frac{a-x}{x}\right)\div\left(\frac{x}{x+a}+\frac{a-x}{x}\right).$

8. $\frac{(a+b)^2-x^2}{2a+2b-2x}\div\frac{(b+x)^2-a^2}{a+b+x}\cdot\left(\frac{a-x+b}{a+b+x}\right)^2.$

9. $\dfrac{\frac{1}{1-a}+\frac{1}{1+a}}{\frac{1}{1+a}-\frac{1}{1-a}}=\dfrac{\frac{2}{1-a^2}}{\frac{-2a}{1-a^2}}=?$

10. $\dfrac{\frac{x-y}{y}+\frac{y}{x+y}}{\frac{1}{x}+\frac{1}{y}}.$

155. Equations. The statement by the use of the symbol of equality (=) that two number expressions are equal is called an **equation.** The number expressions are known as the **left,** or **first member,** and the **right,** or **second member,** according to their position with reference to the equality sign.

There are two classes of equations: (1) **identical equations,** or simply **identities,** (2) **equations of condition, conditional equations,** or simply **equations.**

An **identity** gets its name from the fact that its two members are either identically the same or become the same when the indicated operations are performed. The sign $\equiv$, read "is identically equal to," might be conveniently used in place of the sign = in identities. We have used it in the typeforms. An identity is true for every possible value of the literal numbers involved and is, therefore, of little use in the solution of problems.

The statements: (1) $a+2b=2b+a$, (2) $5=5$, and (3) $(x+y)\ (x-y)=(x^2-y^2)$ are identities. The members of (1) and (2) are exactly the same, and the first member of (3) becomes exactly the same as the second when the indicated multiplication is performed.

An **equation of condition** is an equation involving one or more unknowns which is a true equality only for certain values of these unknowns. The equation $2x-3=7$ is true only when x is 5. The equation $x^2-x=6$ is true only when x is 3 or -2. The equation $x+y=7$ is true when x is 3 and y is 4, when x is 5 and y is 2, or when x is 9 and y is -2, but would not be true if x were 6 and y were 3, or if x were 3 and y were 2.

The four **axioms** of the equation are:

(1) *The same number may be added to both members of an equation without destroying the equality.*

(2) *The same number may be subtracted from both members of an equation without destroying the equality.*

(3) *Each member of an equation may be multiplied by the same number without destroying the equality.*

(4) *Each member of an equation may be divided by the same number (not zero) without destroying the equality.*

How do these four axioms compare with the corresponding axioms of Geometry?

156. Fractional equations.

Exercise 153

Solve the following equations and check:

1. $\frac{3}{5x}+\frac{1}{2x}=\frac{11}{20}$.
2. $\frac{2}{x+3}=\frac{1}{x-2}$.
3. $\frac{x+1}{x-2}=\frac{x-3}{x-5}$.
4. $\frac{4x-3}{2x-1}=\frac{4x-5}{2x-7}$.
5. $\frac{12x-5}{21}=\frac{3x+4}{9x+3}+\frac{4x-5}{7}$.

Hint. Multiplying first by 21 and collecting gives $10=\frac{21x+28}{3x+1}$.

6. $\frac{1}{2}+\frac{2}{x+2}=\frac{13}{8}-\frac{5x}{4x+8}$.
7. $\frac{3x+1}{x-5}=\frac{5x+4}{x+8}-2$.
8. $\frac{x-7}{x+7}-\frac{2x-15}{2x-6}=-\frac{1}{2(x+7)}$.
9. $\frac{2x+1}{2x-1}-\frac{8}{4x^2-1}=\frac{2x-1}{2x+1}$.
10. $\frac{2x+19}{5x^2-5}-\frac{17}{x^2-1}=\frac{3}{1-x}$.
11. $\frac{x+1}{(x+2)\,(x+3)}+\frac{x+2}{(x+1)\,(x+3)}=\frac{x+3}{(x+1)\,(x+2)}$.

Hint. Multiplying by the L. C. D. $(x+1)\,(x+2)\,(x+3)$ gives $(x+1)^2+(x+2)^2=(x+3)^2$

Whence $x^2=4$, $\therefore x=\pm 2$, but -2 does not check because we cannot interpret $\frac{-1}{0}+\frac{0}{-1}=\frac{1}{0}$.

In general, no number can be accepted as a root of an equation if, when it is substituted for the unknown, any denominator reduces to zero. If the numerator of a fraction alone reduces to zero, the value of the fraction is assumed to become zero.

See Chapter XXII for further discussion of Indeterminate Forms.

12. $\dfrac{x-1}{(x-2)\,(x-3)}-\dfrac{x-2}{(x-1)\,(x-3)}=\dfrac{x-3}{(x-1)\,(x-2)}.$

Solution. Clearing of fractions and collecting gives $x^2-8x+12=0$ $\therefore (x-6)\,(x-2)=0$ and $x=6$ or 2. Do both values check?

13. $\dfrac{x-3}{x+1}+\dfrac{x+4}{x-2}=\dfrac{8x+2}{x^2-x-2}+1.$

14. $\dfrac{5}{(x-1)\,(x+2)}-\dfrac{2}{x^2-x-2}=\dfrac{8}{x^2-1}-\dfrac{5}{x^2-4}.$

15. $\dfrac{3-2x}{1-2x}-\dfrac{2x-5}{2x-7}=1-\dfrac{4x^2-2}{7-16x+4x^2}.$

16. $\dfrac{1}{x-1}-\dfrac{1}{2(x+1)}-\dfrac{x+3}{2(x^2+1)}=\dfrac{6}{x^4-1}.$

17. $\dfrac{5x-8}{6x-15}-\dfrac{2x-5}{10x-4}=\dfrac{19x^2-29}{(2x-5)\,(15x-6)}.$

18. $\dfrac{1}{x-8}-\dfrac{1}{x-7}+\dfrac{1}{x-4}=\dfrac{1}{x-5},$

Hint. $\dfrac{1}{x-8}-\dfrac{1}{x-7}=\dfrac{1}{x-5}-\dfrac{1}{x-4},$

$\dfrac{x-7-x+8}{(x-8)(x-7)}=\dfrac{x-4-x+5}{(x-5)(x-4)},\qquad \dfrac{1}{(x-8)(x-7)}=\dfrac{1}{(x-5)(x-4)}.$

19. $\dfrac{7}{x-9}+\dfrac{2}{x-4}=\dfrac{7}{x-7}+\dfrac{2}{x-11}.$

Hint. $\dfrac{7}{x-9}-\dfrac{7}{x-7}=\dfrac{2}{x-11}-\dfrac{2}{x-4}.$

20. $\dfrac{1}{x-13}-\dfrac{2}{x-15}+\dfrac{2}{x-18}=\dfrac{1}{x-19}.$

Solve the following for x:

21. $a+\dfrac{b}{x}=c.$ Ans. $x=\dfrac{b}{c-a}.$

22. $\dfrac{a}{x}+b=\dfrac{b}{x}+a$

23. $\dfrac{x+a}{x-a}=\dfrac{5}{4}.$

24. $\dfrac{a-bx}{ax-b}=\dfrac{3}{4}.$

25. $\dfrac{m+x}{m+n}=\dfrac{m-x}{n-m}.$

26. $\dfrac{x+m}{3}-\dfrac{3}{x+m}=\dfrac{x-m}{3}.$

27. $\dfrac{x+ab}{x-ab}=\dfrac{a^2+ab+b^2}{a^2-ab+b^2}.$

28. $\dfrac{ax}{a+b}+\dfrac{bx}{a-b}=x+\dfrac{2b}{a}.$

29. $\dfrac{2x^2+ax+b}{3x^2+bx+a}=\dfrac{2}{3}.$

CHAPTER XIII

THE FUNCTION, EQUATIONS, AND GRAPHING

157. In an algebraic expression involving several variables it is usually possible to find the value of one of these in terms of the others. It will be recalled that most formulas are written in such form as $i=prt$, $C=2\pi r$, $A=\pi r^2$, . . .

If such an equation as $3x+4y=7$ is solved for x, we have $x=\frac{1}{3}(7-4y)$. If it is solved for y, we have $y=\frac{1}{4}(7-3x)$.

If in an algebraic expression two variables are so related that when a value is given one then a value of the other is determined, the second variable is called a **function** of the first. In $C=2\pi r$, C is a function of r. Its value is fixed when a value is given r. Similarly, A is a function of r in $A=\pi r^2$, and x is a function of y in $x=\frac{1}{3}(7-4y.)$

An algebraic expression involving a single literal number is called a function of that literal number, for the value of the expression is determined by the value given that number. The expression x^3+x^2-x-4 is a function of x because the expression has a definite value for each value given x. The expression y^4-2y^2+4 is a function of y.

When it is necessary to refer to such an expression as x^3+x^2-x-4 several times during a discussion it is customary to represent the function in x by the symbol $f(x)$ which we read as the "f function of x."

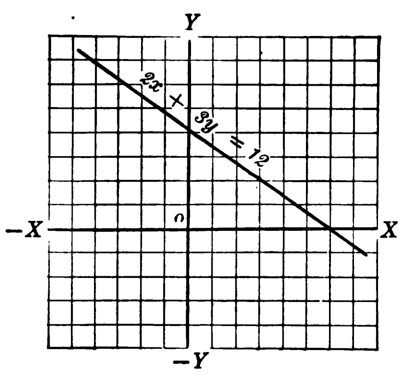
If $f(x)$ is the expression x^3+x^2-x-4, then $f(a)$ in the same discussion will be the expression found by replacing x by a, or a^3+a^2-a-4. That is if $f(x)=x^3+x^2-x-4$, then
$f(a)=a^3+a^2-a-4$ and $f(2)=2^3+2^2-2-4=6$.

Exercise 154

1. If $f(x)=x^2-x+1$, what is $f(y)$?
2. If $f(z)=z^3-3z^2+3z-1$, what is $f(k)$?
3. If $f(y)=y^3-27$, find $f(2)$, $f(0)$, and $f(-3)$.
4. If $f(x)=\dfrac{x-1}{x^2-x+1}$, find $f(1)$, $f(0)$, and $f(-1)$.
5. If $f(x)=x^6-x^4+x^2-1$, find $f(\frac{1}{2})$, and $f(-\frac{1}{2})$.
6. If $f(n)=n^3-3n^2-3n+11$, find $f(0)$, $f(-1)$, $f(-2)$, and $f(-\frac{1}{2})$.
7. If $f(a)=a^3+3a^2+3a+2$, what is $f(-2)$? Therefore what binomial is a factor of $f(a)$ by the Factor Theorem?
8. If $f(x)=x^3-6x^2+11x-6$, show that $f(3)=f(2)=f(1)=0$, and name the three prime factors of $f(x)$.
9. If $f(x)=x^2-2x+1$, find $f(a-1)$, $f(a+1)$, and $f(a+y)$.

158. An equation of the first degree in two unknowns. It will be recalled (§ 63) that the **degree of a term** is determined by the number of literal prime factors that it contains provided that none of these appear in a denominator. The term $2a^2b$ is of the third degree. It is of the first degree in b and the second degree in a.

The **degree** of a **polynomial** with respect to, or in a given letter, is determined by its term of highest degree. The polynomial $a^3+3a^2b+3ab^2+b^3$ is of the third degree in a as well as in b.



It will be recalled that the graph of an equation of the first degree in two variables is a straight line. This explains the common name, "**linear equation.**" Note the graph of the equation $2x+3y=12$ in the accompanying figure.

If we express x as a function of y, we have $x=\frac{12-3y}{2}\cdot$ For every possible value of y there is a value for x. Similarly we may express y as a function of x, and show that for every possible value of x there is a value of y. The graph is the locus of a point under the condition imposed by the relation of x to y or of y to x. (See § 84.)

The equation $2x+3y=12$ is a true equality only under the condition imposed by this relation. It will be proved in analytic geometry that the locus of every equation of the first degree in two unknowns is a straight line.

159. An equation of the first degree in three or more unknowns.

As with the linear equation, so also with an equation of the first degree in three or more unknowns, there is a limitless number of sets of values for the unknowns that satisfy the equation.

Take the equation $3x+2y-z=5$ and note that each of the following sets of values satisfies it:

when $x=1$,	1,	1,	2,	2,	2,	2,	3,	3,
and $y=1$,	2,	3,	1,	-2,	0,	2,	3,	4,
then $z=0$,	2,	4,	3,	-3,	1,	5,	10,	12.

An equation of the first degree in two or more unknowns has a limitless number of solutions, and is therefore said to be **indeterminate.**

160. Two or more equations involving the same two or more unknowns are said to form a **system of equations.** We shall first discuss

Linear systems. If two equations of the first degree in the same two unknowns have no common pair of values they are said to be **incompatible,** or **contradictory,** and, if graphed, their straight lines will be found to be parallel such

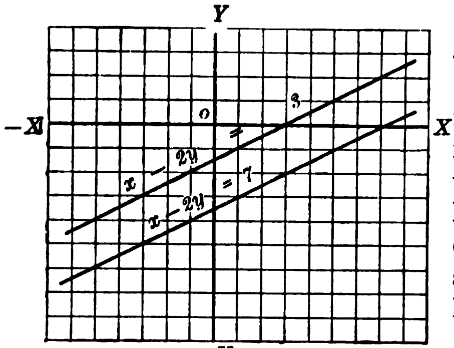


Fig. 1

as $x - 2y = 3$ and $x-2y=7$. (Fig. 1.)

Some systems have many pairs of values in common. When these are graphed their lines will be found to coincide, as $x-2y=5$ and $3x-6y=15$ in Fig. 2.

The second equation may be obtained from the first by multiplying both members by 3 and is called a **dependent equation.**

Two equations are said to be **independent** when one cannot be obtained from the other by any process that does not destroy the relation of the unknowns.

Two **independent linear equations** that have a single set of values in common, are said to form a **simultaneous linear system.** Their graphs will intersect in one point since two straight lines on a plane that are not parallel intersect in but one point. (See Fig. 3 where $2x+3y=6$ and $3x-y=5$ form such a system.)

161. The solution of a simultaneous linear system is completed when the common set of values for the unknowns is found. This may be accomplished by one of the following methods.

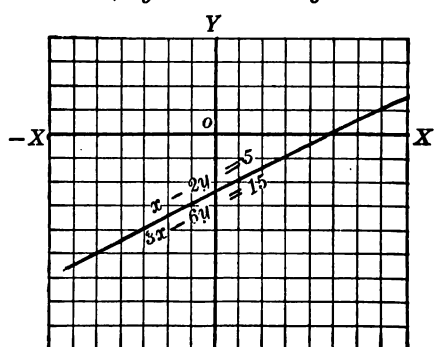


Fig. 2

Method I. Elimination of one of the unknowns by addition or subtraction. (See § 87.)

Rule. *Multiply the members of both equations by such numbers as will give equal absolute values for the coefficients of one of the unknowns. Add or subtract the resulting equations to eliminate that unknown and solve the resulting equation for the other unknown. Complete the solution.*

Method II. Elimination of one of the unknowns by substitution. (See § 88.)

Rule. *Solve one of the equations for one unknown in terms of the other. Substitute the result for that unknown in the other equation and complete the solution.*

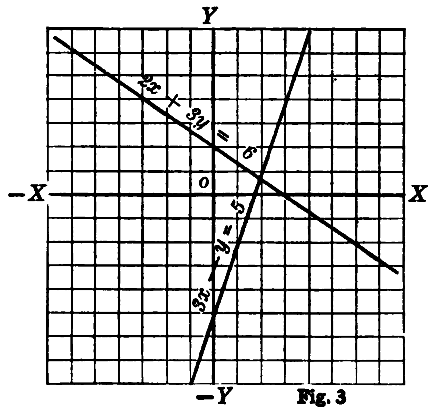


Fig. 3

Method III. Elimination by comparison.

Rule. *In each equation find the value of the same unknown in terms of the other. Equate these two values and solve the resulting equation.*

Illustrative example. Given $2x+4y=12$ (1)
and $3x-2y=10$ (2)

From (1) $x=6-2y$ and from (2) $x=\frac{10+2y}{3}$

therefore $6-2y=\frac{10+2y}{3}$. Solving, gives $y=1$ and $x=4$.

Exercise 155

Solve the following systems by Method I and check:

1. $5y-2x=6$
$8y-5x=-3$.

2. $5x+2y=19$
$2x-5y=-4$.

3. $2a+3b=1$
$3a-2b=21.$

4. $5m+3n=6$
$3m-5n=7.$

5. $5x+11y=13$
$3x-7y=1.$

6. $x=\frac{1}{3}(9-y)$
$y=\frac{1}{2}(x+4).$

7. $x+1=\frac{1}{2}(x+y)$
$y-1=2(x-y).$

8. $9x=11+3y$
$6y=x-5.$

Solve the following systems by Method II and check:

9. $3R-4r=20$
$7R+11r=6.$

10. $5a-2b=3$
$4b-4a=-3.$

11. $7m-2n=\frac{19}{10}$
$5m+3n=\frac{1}{4}.$

12. $\frac{1}{2}x-\frac{1}{3}y=6$
$\frac{1}{4}x+\frac{1}{6}y=0.$

In solving such a system as is given in No. 12, any one of the three methods given for the solution of a simultaneous system in two unknowns may be employed, but the first method will be found most satisfactory.

Solve the following by Method III and check:

13. $\frac{1}{2}y-3x=2$
$y=14x.$

14. $5y=2x+1$
$8y=5x-11.$

15. $\frac{x}{2}+\frac{y}{3}-7=0$
$\frac{x}{3}+\frac{y}{2}-8=0.$

16. $3x-5y=4$
$5y-2x=\frac{-3}{2}.$

Graph both lines in each of the following sets, and locate carefully their point of intersection. Verify the accuracy of your graphs by finding the solution (if there is one) by Method I:

17. $3x-5y=1$
$4y+2x=8.$

18. $2x+3y=8$
$3x+\frac{9y}{2}=12$

19. $\frac{x}{2}+2y=3$
$3x+12y=4.$

20. $2x+5y=-12$
$3x-2y=1.$

21. $5x-y=4$
$2x+4y=-5.$

22. $3x-2y=-1$
$6x+8y=16.$

Solve the following systems by any method and check:

23. $x-\dfrac{7y}{5}=-23$
$x+\dfrac{y}{5}=17.$

24. $\dfrac{x+3}{2}+5y=9$
$\dfrac{y+9}{10}-\dfrac{x-2}{3}=0.$

25. $0.3x+0.2y=9.5$
$0.2x+0.3y=10.5.$

26. $.02x-.03y=60$
$.03x+.02y=155.$

Solve the following for x and y and check:

27. $x+y=2a$
$(a-b)x=(a+b)y.$

28. $cx-by=0$
$bx+cy=b^2+c^2.$

29. $x+my+m^2=0$
$x+ny+n^2=0.$

30. $x+ay=-1$
$y+c(x+1)=0$

31. $ax-by=a^2+b^2$
$(a-b)x+(a+b)y=2(a^2-b^2).$

While the following are not linear systems, they are simultaneous and can be solved conveniently by any of the three methods.

Do not clear of fractions until one unknown is eliminated. Solve each and check:

32. $\dfrac{10}{x}-\dfrac{9}{y}=8$
$\dfrac{8}{x}+\dfrac{15}{y}=-1.$

33. $\dfrac{3}{2x}-\dfrac{1}{2y}=7$
$\dfrac{4}{3x}-\dfrac{3}{2y}=2.$

34. $\dfrac{1}{x}-\dfrac{1}{y}=m$
$\dfrac{1}{x}+\dfrac{1}{y}=n.$

35. $\dfrac{a}{bx}+\dfrac{b}{ay}=a+b$
$\dfrac{b}{x}+\dfrac{a}{y}=a^2+b^2.$

162. Solution of a simultaneous linear system by determinants.

Let $a_1x+b_1y=c_1$ (1)
and $a_2x+b_2y=c_2$ (2) be a simultaneous linear system.

Multiplying both members of (1) by b_2 and both members of (2) by $-b_1$, we will have

$$a_1b_2x+b_1b_2y=b_2c_1 \quad (3)$$
$$-a_2b_1x-b_1b_2y=-b_1c_2 \quad (4).$$

Adding (3) and (4) and collecting gives

$(a_1b_2-a_2b_1)x=b_2c_1-b_1c_2$

and $x=\dfrac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}$, if a_1b_2 is not equal to $(\neq)a_2b_1$.

Similarly, multiplying both members of (1) by $-a_2$ and of (2) by a_1 and adding gives

$(a_1b_2-a_2b_1)y=a_1c_2-a_2c_1$ and $y=\dfrac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$, if $a_1b_2\neq a_2b_1$.

The denominators of the fractional values of x and y are the same and may be written compactly in the form $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, which is called a **determinant.** It is said to be of the second order since it has two rows and two columns. The letters a_1, a_2, b_1, and b_2 are called the **elements** of the **determinant** and a_1b_2, its **principal diagonal.** The value of a determinant of the second order is found by subtracting from the product of the elements that form its principal diagonal, the product of the other two elements.

Similarly, the numerators of the fractional values of x and y may be written as determinants, that for x being $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and for y being $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.

The values of x and y may be written in determinant symbols and are $x=\dfrac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$, and $y=\dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$

The numerator of the value of x is obtained from the denominator by substituting for the column $\begin{vmatrix} a_1 \\ a_2 \end{vmatrix}$, which are the coefficients of x in the given equations (1) and (2), the column of the known terms $\begin{vmatrix} c_1 \\ c_2 \end{vmatrix}$. Similarly, the numerator of

the value of y is obtained from the denominator by replacing the column $\begin{vmatrix} b_1 \\ b_2 \end{vmatrix}$ by $\begin{vmatrix} c_1 \\ c_2 \end{vmatrix}$.

The following **illustrative examples** may serve to show the possibilities arising from the use of determinants:

1. Solve $2x+4y=14$
$3x+\ y=11.$

$$x=\frac{\begin{vmatrix} 14 & 4 \\ 11 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}}=\frac{14-44}{2-12}=\frac{-30}{-10}=3.$$

$$y=\frac{\begin{vmatrix} 2 & 14 \\ 3 & 11 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}}=\frac{22-42}{-10}=\frac{-20}{-10}=2.$$

2. $x-3y=6$
$4x-5y=24.$

$$x=\frac{\begin{vmatrix} 6 & -3 \\ 24 & -5 \end{vmatrix}}{\begin{vmatrix} 1 & -3 \\ 4 & -5 \end{vmatrix}}=\frac{-30+72}{-5+12}=\frac{42}{7}=6.$$

$$y=\frac{\begin{vmatrix} 1 & 6 \\ 4 & 24 \end{vmatrix}}{7}=\frac{24-24}{7}=\frac{0}{7}=0.$$

Exercise 156

Evaluate the following determinants:

1. $\begin{vmatrix} 3 & -3 \\ 2 & 4 \end{vmatrix}.$ Ans. 18. **2.** $\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix}.$

3. $\begin{vmatrix} 6 & -2 \\ 4 & -3 \end{vmatrix}.$ **4.** $\begin{vmatrix} -5 & -3 \\ -6 & -2 \end{vmatrix}.$ **5.** $\begin{vmatrix} a & c \\ b & d \end{vmatrix}.$

6. $\begin{vmatrix} 2a & -4b \\ 3b & 7a \end{vmatrix}.$ **7.** $\begin{vmatrix} a+b & -b \\ b & a-b \end{vmatrix}.$

8. $\begin{vmatrix} 3a^2 & 2ab \\ 7ab & 4b^2 \end{vmatrix}.$ **9.** $\begin{vmatrix} 1 & x+y \\ x-y & y^2 \end{vmatrix}.$

10. $\begin{vmatrix} 2a+3b & a-b \\ 3a-2b & 2a-b \end{vmatrix}.$

Solve each of the following systems by determinants:

11. $2x-3y=-10$
$3x+5y=\ \ 4.$

12. $5x+11y=13$
$3x-\ 7y=\ 1.$

13. $2x+5y=-14$
$3x-7y=\quad 8.$

14. $3x-5y=7$
$5x+3y=6.$

15. $5x+9y=28$
$7x+3y=20.$

16. $8x+9y=26$
$32x-3y=26.$

17. $21x-23y=\ 2$
$7x-19y=12.$

18. $\frac{3}{2}x-7y=-38$
$\frac{3}{2}y-7x=-72.$

19. $\frac{1}{6}x-\frac{1}{8}y=-1$
$\frac{1}{6}y-\frac{1}{4}x=-2.$

20. $5x+7y=49$
$7x+5y=47.$

163. Systems of equations in more than two unknowns. We have already found that such an equation as $3x+2y-z=5$, (1), has a limitless number of sets of values that satisfy it. (See § 159.) It will be shown in analytic geometry that the locus of such an equation, with reference to three axes, x, y, and z, each perpendicular to the other two, is a plane in space. Such an equation as $3x+2y-z=8$, (2), has no set of values in common with (1) and they are said to be **incompatible.** The locus of (2) is a plane parallel to that of (1). If, however, we take such an equation as $2x-3y+z=7$, (3), (1) and (3) will be found to have many sets of values in common, for adding (1) and (3) gives $5x-y=12$, which is a linear equation and has a limitless number of sets of values.

Equations (1) and (3) are said to be **independent equations** since neither can be obtained from the other by any process that does not destroy the relation among the unknowns.

Three independent equations in the same three unknowns that form a **simultaneous system** have but a single set of values that are common for all three. We must have as many independent equations as there are unknowns if we are to find the solution of the system.

The system
$$3x+2y-z=\ \ 5 \qquad (1)$$
$$2x-3y+z=\ \ 7 \qquad (3)$$
and $$x+\ y+z=-3 \qquad (4)$$
may be solved by any one of the three methods given in § 161 but Method I is

very convenient, for adding (1) and (3) gives $5x-y=12$ (5) and adding (1) and (4) gives $4x+3y=2$ (6).

Equations (5) and (6) form the simultaneous linear system
$5x-\ \ y=12$ (5)
$4x+3y=\ \ 2$ (6) which may be solved by Method I by multiplying (5) by 3, and adding the product to (6) gives $19x=38$, or $x=2$. Therefore $y=-2$ and $z=-3$.

Exercise 157

Solve the following simultaneous systems:

1. $$\begin{aligned} 3x-2y+\ z&=\ \ 8\\ 2x-3y-2z&=-1\\ x+\ y+\ z&=\ \ 3. \end{aligned}$$

2. $$\begin{aligned} 3x-\ 4y+\ 6z&=\ 60\\ 4x-\ 5y+\ 4z&=\ 20\\ 10x-12y+15z&=180. \end{aligned}$$

3. $$\begin{aligned} 3x+5y+6z&=31\\ 3x+4y+5z&=26\\ 2x+3y+4z&=20. \end{aligned}$$

4. $$\begin{aligned} x+y&=5\\ y+z&=7\\ x+z&=9. \end{aligned}$$

5. $$\begin{aligned} a+2b-3c&=\ 6\\ 2a+4b-7c&=\ 9\\ 3a-\ b\qquad&=23. \end{aligned}$$

6. $$\begin{aligned} 7x+y\ &=4z+w\\ x+w\ &=y\\ 2z+3y\ &=15+w\\ 3y+8\ &=7x+2z+3w. \end{aligned}$$

7. $$\begin{aligned} 2x-y&=\ 8\\ 3y-z&=13\\ 4z-w&=16\\ 5w-x&=13. \end{aligned}$$

8. $$\begin{aligned} x+y+z&=6\\ y+z+w&=9\\ x+z+w&=8\\ x+y+z+w&=10. \end{aligned}$$

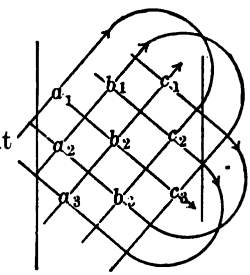
9. $$\begin{aligned} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}&=24\\ \frac{1}{x}+\frac{1}{y}&=14\\ \frac{2}{x}-\frac{1}{z}&=2. \end{aligned}$$

10. $$\begin{aligned} \frac{1}{x}-\frac{2}{y}+3&=0\\ \frac{1}{y}-\frac{3}{z}+4&=0\\ \frac{1}{z}-\frac{4}{x}+2&=0. \end{aligned}$$

164. Simultaneous systems involving three or more unknowns may be solved very conveniently by determinants obtained in the same manner as were those of § 162. A

system with three unknowns will involve a determinant of the third order with three rows and three columns; one with four unknowns, a determinant of the fourth order. Determinants above the third order are best studied in a more advanced course.

A **determinant** of the **third order** may be defined as a compact method for indicating six products of three factors each, three of which are positive and three negative.

Determinant 

$$= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

The arrows indicate how the products are obtained and their character.

Illustrative example.

1. Solve
$$\begin{aligned} 2x+3y+\ z&=11\\ x+2y+3z&=14\\ 3x+\ y+2z&=11. \end{aligned}$$

$$x=\frac{\begin{vmatrix} 11 & 3 & 1\\ 14 & 2 & 3\\ 11 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2 \end{vmatrix}},\quad y=\frac{\begin{vmatrix} 2 & 11 & 1\\ 1 & 14 & 3\\ 3 & 11 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2 \end{vmatrix}},\quad z=\frac{\begin{vmatrix} 2 & 3 & 11\\ 1 & 2 & 14\\ 3 & 1 & 11 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2 \end{vmatrix}}.$$

Note that the determinant denominators are the same for **all three** unknowns and are the columns of the coefficients of x, y, and z in their order. Note that the numerators are obtained from the denominators as in § 162 by replacing the column of the coefficients of x, y, or z by the column of the constants $\begin{vmatrix} 11\\ 14\\ 11. \end{vmatrix}$

The value of the common determinant denominator is obtained by following the terms of the definition of a determinant of the third order.

$$\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix} = 2\cdot2\cdot2+1\cdot1\cdot1+3\cdot3\cdot3-3\cdot2\cdot1$$
$$-1\cdot3\cdot2-2\cdot1\cdot3=8+1+27-6-6-6=18.$$

The determinant numerator for $x=11\cdot2\cdot2+14\cdot1\cdot1+11\cdot3\cdot3$
$$-11\cdot2\cdot1-14\cdot3\cdot2-11\cdot1\cdot3=44+14+99-22-84-33=18$$
$$\therefore x=\frac{18}{18}=1. \text{ Similarly, } y=\frac{36}{18}=2 \text{ and } z=\frac{54}{18}=3.$$

Exercise 158

Evaluate the following determinants:

1. $\begin{vmatrix} 2 & 1 & 5 \\ 3 & 2 & 3 \\ 1 & 4 & 2 \end{vmatrix}$. Ans. 31.

2. $\begin{vmatrix} 5 & 1 & 6 \\ 3 & 4 & 2 \\ 2 & 3 & 3 \end{vmatrix}$.

3. $\begin{vmatrix} 2 & 4 & -2 \\ 0 & 3 & 3 \\ -1 & -5 & 0 \end{vmatrix}$.

4. $\begin{vmatrix} 5 & 10 & -3 \\ 3 & 6 & 2 \\ 2 & 4 & -8 \end{vmatrix}$.

5. $\begin{vmatrix} a & 3 & -3 \\ b & 2 & 4 \\ 2a & -1 & 2 \end{vmatrix}$.

6. $\begin{vmatrix} a & b & b \\ b & a & a \\ a & b & a \end{vmatrix}$.

Solve the following simultaneous equations by determinants:

7. $2x+3y-2z=2$
$3x-y+z=4$
$x+2y-3z=-4.$

8. $x+y+z=6$
$2x-y-z=3$
$3x+4y+3z=16.$

9. $2a-b+c=12$
$3a+2b-2c=-10$
$5a+3b+c=6.$

10. $x+y+z=-5$
$2x+3y-3z=5$
$3x-2y+4z=-18.$

11. $2x+3y+4z=19$
$5x-2y-3z=4$
$7y+5z=0.$

12. $4x+3y=3$
$6y-2z=7$
$5x+3z=-5.$

165. Solution of problems.

Exercise 159

Some of the problems of algebra may be classified according to the essential feature on which it is best to build the necessary equations.

I. **Work problems.** Essential feature: what fraction of the whole work is completed in some unit of time?

1. If A works alone, he can lay a cement walk in 10 days. If B works with him, they can lay the walk in 6 days. How many days would it take B working alone?

Let x represent the number of days that B would require if working alone.

In one day A can do $\frac{1}{10}$ of the work and B, $\frac{1}{x}$ of the work and both together will complete $\frac{1}{10}+\frac{1}{x}$ of the whole.

But according to the problem they will finish $\frac{1}{6}$ of the walk in one day, therefore the equation, $\frac{1}{10}+\frac{1}{x}=\frac{1}{6}$.

2. Two men if they work together will finish a certain task in 10 hours. If, at the end of the 6th hour, one man is withdrawn the other will finish 12 hours later. How many hours would each require if working alone?

The equations are $\frac{1}{x}+\frac{1}{y}=\frac{1}{10}$ (1)

and $\frac{6}{x}+\frac{6}{y}+\frac{12}{y}=1$ (2).

Can you explain (1) as in No. 1 and (2) by the axiom that the whole is equal to the sum of all of its parts?

3. Water may enter a tank through three pipes A, B, and C. If the tank is empty and all the pipes are opened, the tank will be filled in $2\frac{2}{3}$ hours. If A and B alone are opened, it will be filled in $3\frac{3}{7}$ hours. If B and C alone are opened, it will be filled in $4\frac{4}{5}$ hours. How many hours would each pipe require if opened alone?

The first equation is $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2\frac{2}{3}}=\frac{3}{8}$.

4. A tank has one outlet pipe and two inlets. If the tank is empty and all are opened, it will be filled in 12 hours. If the outlet is closed, the two inlets will fill it in 6 hours. If the smaller inlet is closed and the outlet opened the larger inlet will fill the tank in 60 hours. If the tank is full and the inlets closed how long will it take the outlet to empty it? If empty and the outlet closed, how long would it take each inlet alone to fill it? What would happen if the tank were full and the outlet and the smaller inlet were opened?

II. **Mixture problems.** Essential feature: what part, or per cent, some element is of the whole mixture.

5. How much salt must be added to a 2% salt solution weighing 20 lbs. to make it a 5% solution?

The equation is $\frac{2}{100}(20)+x=\frac{5}{100}(20+x)$.

Can you explain it?

6. A druggist has an acid in two strengths, one 70% pure and the other 95% pure. How much of each must he take to make 20 liters 85% pure?

The equations are $x+y=20$ and $\frac{70}{100}x+\frac{95}{100}y=\frac{85}{100}20$.

7. How many quarts of milk testing 4% butter fat must be added to 20 quarts of cream testing 22% butter fat to make the cream test 16%?

The equation is $\frac{4}{100}x+\frac{22}{100}20=\frac{16}{100}(x+20)$.

III. **Digit problems.** Essential feature: if letters are used for digits the place of the digit must be accounted for by multiplying by 10, 100 . . . . .

8. The sum of the digits of a two-digit number is 12 and, if the digits were interchanged, the resulting number would be 18 more than the given number. What is the number?

The equations are $x+y=12$ and $10x+y+18=10y+x$.

Explain these.

9. The sum of the digits of a three-digit number is 14 and the units' digit is equal to the sum of the other two. If the tens' and hundreds' digits were interchanged, the number would be increased by 270. What is the number?

The equation from the last condition is $100x+10y+z+270=100y+10x+z$. Can you explain?

IV. **Uniform motion problems.** Formula, $d=tr$.

10. A freight train running 30 miles per hour is 50 miles ahead of a passenger train running 40 miles per hour. In how many hours will the passenger train overtake the freight train? Solve by a graph before solving by an equation.

11. A train running 30 miles per hour requires 20 minutes longer to go a certain distance than a train running 40 miles per hour. Find the distance.

12. Two automobiles start together in the same direction around a circular track. One can make the circuit in $2\frac{1}{2}$ minutes and the other in 3 minutes. In how many minutes will they be together again?

Hint. The faster machine must gain one circuit and the equation is $\frac{x}{2\frac{1}{2}}-\frac{x}{3}=1.$

V. **Lever problems.** Essential feature, Law of Levers, $Lw=lW$, or the weight on one arm times its distance from the fulcrum equals the weight on the other arm times its distance from the fulcrum.

13. Assuming that the bar of the lever itself has no appreciable weight, how far from the fulcrum on one side must a weight of 100 lbs. be placed to balance a weight of 80 lbs. placed five feet from the fulcrum on the other side?

The equation is $5 \cdot 80=100x$. Explain by a figure.

14. A lever 16 feet long, with its fulcrum at the center, has a weight of 12 lbs. hung at one end and 8 lbs. hung at the other. Where must an additional weight of 8 lbs. be hung to provide an exact balance?

At the first reading of a problem the student should ascertain (1) if it belongs to some particular group, (2) how many unknowns are involved, (3) if there is sufficient data for the necessary equations.

15. How much cream testing 16% butter fat must be added to 40 quarts of milk testing 3% butter fat to make the milk pass the legal test of 4% butter fat?

16. A works three-fourths as fast as B and both together can finish a task in $5\frac{1}{7}$ hours. How long would it take each if he works alone?

17. A certain government has two kinds of old coins, one 95% silver and 5% copper, the other 80% copper and 20% silver. How much of each must be used to make 2000 lbs. of metal for coinage 92% silver?

18. The sum of the digits of a two-digit number is 9 and if the digits were interchanged, the number would be increased by 27. What is the number?

19. A lever 12 feet long, having a weight at each end, is balanced at a point $4\frac{1}{2}$ feet from one end. If the weight on the shorter arm is 100 lbs., what is the weight on the longer arm?

20. A man invests $12,000 in two kinds of bonds, a part in 6% bonds at 92 and the rest in 5% bonds at 70. If his annual interest is $800, what is his investment in each kind?

Hint. If x is the number of dollars that he invests in the first kind, then $x/92$ is the number of those bonds that he buys and $6x/92$ is his annual interest from them.

21. A merchant has an acid in two strengths such that 10 quarts of the first kind and 8 of the second makes a mixture 80% pure, while 7 quarts of the first and 2 of the second makes a mixture 84% pure. What is the % of purity of each kind?

22. A man invests a certain sum in 5% bonds at 90 and twice as much in 4% bonds at 80. If his annual interest from the investment is $560, what sum did he put into each kind?

23. How much pure copper must be added to 500 lbs. known to be 90% copper to make a metal 96% pure?

24. How much tin must be added to 2000 lbs. of metal testing 92% copper and 8% tin to make the mass 88% copper?

25. A and B can do a task in $5\frac{5}{11}$ hours, that A and C could do in $6\frac{2}{3}$ hours, and B and C in $7\frac{1}{2}$ hours. How many hours would it take each if he worked alone?

26. A camp equipment of 90 lbs. is swung on a pole between two boys. The boys are 10 feet apart and the weight is suspended 4 feet from one boy. What weight does each carry?

27. The sum of the digits of a three-digit number is 10. If the tens' and units' digits are interchanged the number is increased by 27. If the units and hundreds digits are interchanged it is increased by 198. What is the number?

28. A man invests a certain part of a sum of money in 5% bonds at 90 and the rest in 4% at 80 and his annual interest is $992. If the first part had been invested in the 4% bonds at 80 and the rest in the 5% bonds at 90, his interest would have been $984. What amount does he invest in each?

29. A messenger starts from a camp at the rate of 10 miles per hour, and 15 minutes later a second messenger starts after the first at the rate of 15 miles per hour. In what time will the second overtake the first?

30. If a number of two digits is divided by the sum of its digits the quotient is 7 and the remainder 3. What is the number if the digit in units' place is 3 less than the digit in tens' place?

31. A certain number is added to each of the numbers 2, 9, 3, and 7. If the product of the first two results equals the product of the second two, what is the number that is added?

32. A country grocer sells Mrs. Brown 3 pounds of butter and 4 dozen eggs for $3.60. He buys from Mrs. Jones 4 pounds of butter and 5 dozen eggs for $4.00. Find the grocer's

retail price on each if he makes a profit of 10 cents a pound on butter and 5 cents a dozen on eggs.

33. A grocer wishes to mix 60 lbs. of coffee at a total cost of $16.00. If he uses two grades of coffee, one costing 36 cents and the other 20 cents a pound, how many pounds of each must he use?

34. The sum of the digits of a two-digit number is 12. The quotient of the number divided by the units' digit is 6. What is the number?

35. A chemist has an acid in two strengths, one 85% pure and the other 60% pure. He has an order for 15 ounces of the acid 75% pure. How many ounces of each kind must he use to fill the order?

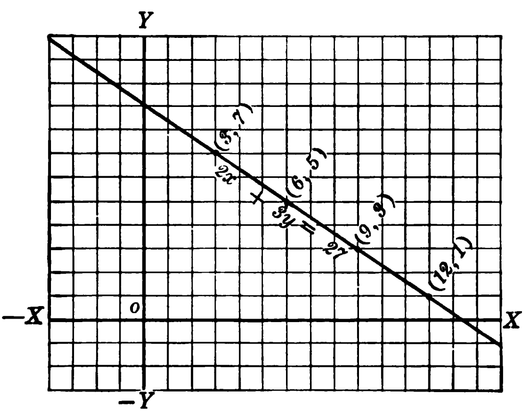
166. Applications of the indeterminate equation.

Exercise 160

1. Find all positive integral sets of values for $2x+3y=27$.

Expressing x as a function of y, $x=\frac{27-3y}{2}=13-y+\frac{1-y}{2}$.

Now if the value of x is to be an integer such values of y must be used as will make $\frac{1-y}{2}$ zero or an integer. If $y=1$, the fraction becomes zero. If $y=3$, it becomes -1. Therefore when $y=1$, 3, 5, 7, then $x=12$, 9, 6, 3.



There are four sets of positive integral values. A carefully constructed graph will pass through the four points as indicated in the figure.

2. In how many ways can a debt of $89 be paid with $2 and $5 bills?

3. A man wishes to pay a bill of $2.23 in cents, dimes, and quarters using the same number of cents as of dimes. In how many ways can he do this?

4. A child's bank is found to contain $2.84 in cents, nickels, and dimes. If there are 60 coins in the bank, in how many ways can the amount be made up?

5. A farmer makes a shipment of chickens, ducks, and turkeys, 45 fowls in all. The shipment nets him $120, the chickens netting $1.00 each, the ducks $2.50, and the turkeys $4.50. In how many ways may the shipment have been made up?

6. A wholesale grocer has coffee in one pound cans which he sells at 25 and 35 cents, respectively. He fills an order for $22.55 made up of cans from both varieties. In how many ways can he do this? How many of each kind would he have shipped if the purchaser had requested that the number of each be approximately the same?

CHAPTER XIV

EXPONENTS, RADICALS, AND IMAGINARIES

167. Exponents. Define each of the following: power, root, square root, cube root, term, like or similar terms.

We have defined an **exponent** as a number so placed as to indicate how many times another number is to be taken as a factor. All exponents so far in our study of algebra have been positive integers or, when literal numbers, have been assumed to be positive integers. It is the purpose of this chapter to expand our knowledge of exponents to include the whole field of real numbers, i.e., zero, negative numbers, and fractions, as well as positive integers.

We have learned and used the laws of exponents (see pages 68, 197, and 200), which were conveniently stated in the following **typeforms** or **formulas**:

I. For multiplication, $a^m \cdot a^n \equiv a^{m+n}$ and $(a^m \cdot b^n)\ (a^x \cdot b^y) \equiv a^{m+x}b^{n+y}$.

II. For division, $a^m \div a^n \equiv a^{m-n}$ and $(a^m b^n) \div (a^x b^y) \equiv a^{m-x}b^{n-y}$.

III. For finding a power, $(a^m)^n \equiv a^{mn}$, and $(a^m b^n)^x \equiv a^{mx}b^{nx}$.

The typeform for finding a root has not been given previously, but the rule may be found on page 200 and the typeform will be made evident by the discussion and use of fractional exponents in this chapter.

IV. For finding a root, $\sqrt[n]{a^m} \equiv a^{\frac{m}{n}}$, and $\sqrt[x]{a^m b^n} \equiv a^{\frac{m}{x}} b^{\frac{n}{x}}$.

For convenience we repeat the proof of the rule for finding the exponent in multiplication. It is as follows:

$a^m \cdot a^n \equiv (a \cdot a \cdot a \cdot a \ldots$ to m factors$) \cdot (a \cdot a \cdot a \cdot a \ldots$ to n factors$) \equiv a \cdot a \cdot a \cdot a \cdot a \cdot a \ldots$ to $m+n$ factors, or a^{m+n}.

State the rule for finding the exponent of a letter in the product of several terms involving different literal factors.

Make the proof for it similar to that just given for $a^m \cdot a^n$. Similarly, prove both typeforms for division and state the rules.

Prove the typeforms for finding a power and state the rules.

Exponents are very convenient for writing compactly very large numbers. For instance it is said that the cost of the Great War to the world was \$200,000,000,000. This may be written \$$2 \cdot 10^{11}$.

The "light year" of the astronomer is the distance that a ray of light will travel in one year, or $365 \cdot 24 \cdot 60 \cdot 60 \cdot 186000$ miles. (A ray of light travels 186000 miles in a second of time.) This distance is a little less than $58657 \cdot 10^8$ miles. Check by actual multiplication.

Exercise 161

Write at sight the result for each of the following:

1. $(a^5)^3$. **2.** $(a^m)^2$. **3.** $(b^5)^x$. **4.** 10^7.

5. $(m^a)^b$. **6.** $(a^{mx})^2$. **7.** $(a^{mx})^n$. **8.** $(a^m)^m$.

9. $(x^{2a})^{3a}$. **10.** $(a^m)^m \cdot (a^n b^x)$. **11.** $(a^2 b^n) \cdot (a^n b^2)$.

12. $(3^2 \cdot 2^3) \cdot (3^3 \cdot 2^2)$. **13.** $(2^2 \cdot 3^3 \cdot 5^5)^2$. **14.** $(a^x b^x)^n \cdot (a^n b^n)^x$.

15. $(a^5 b^4) \div (a^4 b^3)$. **16.** $(3ab^2c^3) \div (abc^2)$.

17. $(5^5 \cdot 3^4 \cdot 2^3) \div (5^2 \cdot 3^2 \cdot 2^3)$. **18.** $(24a^n b^n) \div (3a^2 b^3)$.

19. $(a^4 b^3 c^2) \div (a^2 b^2 c) \cdot (abc^2) \div (a^2 bc^2)$.

20. $(x^3 y^2 z) \div (xyz) \cdot (x^2 yz^2) \div (xy^2 z) \cdot (x^2 yz)$.

21. $(x^a b^y)^2 \cdot (x^a b^y)^3 \div (x^2 b^3)^a \cdot (x^3 b^2)^y$.

168. The zero exponent and its meaning.

According to the typeform for division, $a^3 \div a^3 = a^0$. Similarly, $a^m \div a^n$ gives a^0 when $m = n$. If the division of a^3 by a^3 is performed in fractional form the meaning of a^0 is evident for $\frac{a^3}{a^3} = 1$. $\therefore a^0 = 1$. Or, by the typeform for multiplication $(a+b)^m \cdot (a+b)^0 = (a+b)^m$. But $(a+b)^m \cdot 1 = (a+b)^m$. $\therefore (a+b)^0 = 1$. Hence we have the **principle:**

Any number expression whose exponent is 0 is numerically equal to 1.

Exercise 162

Simplify each of the following:

1. $a^2 \cdot a^0 \cdot a^3$. 2. $a^2 \cdot b^0 \cdot c^3 \cdot d^0$. 3. $(2a)^2 \cdot (2a)^0 \cdot (2a)^3$.

4. $(a^x b^y c^z d^w)^0$. 5. $(ax-by)^2 \cdot (ax-by)^0 \cdot (ax-by)^3$.

Find the numerical value for each of the following:

6. $5^2-4^0+3^2 \cdot 3^0-2^0 \cdot 2^3+2^2$.

7. $(4a-2)^0+7 \cdot 2^0-(7 \cdot 2)^0-(6-3a)^0+4(4^2-3^3)^0$.

8. $(\frac{1}{2})^2-(\frac{3}{2})^0+4 \div (\frac{1}{2})^2 \cdot 2^0 \div (2 \cdot 3^2 \cdot 4^3 \div 8^2)^0+(4\frac{3}{8}-2\frac{1}{2})^0$.

169. The negative exponent and its meaning.

From the typeform for division we have $a^2 \div a^3 = a^{-1}$ and, in general, $a^m \div a^n = a^{m-n}$ gives a negative exponent when m is less than n.

When the division is placed in the form of a fraction and reduced to its simplest form, the result is as follows:

$\frac{a^2}{a^3} = \frac{1}{a}$. (Dividing both members by a^2.) Evidently, since $a^2 \div a^3$ gives a^{-1} and $\frac{a^2}{a^3}$ gives $\frac{1}{a}$, then $a^{-1} \equiv \frac{1}{a}$. In general $a^x \div a^{x+y} = a^{-y}$ and $\frac{a^x}{a^{x+y}} \equiv \frac{a^x}{a^x \cdot a^y} = \frac{1}{a^y}$. $\therefore a^{-y} \equiv \frac{1}{a^y}$.

Hence the **principle**:

The minus sign before an exponent indicates that the reciprocal of the expression affected by the exponent is to be taken with the exponent positive.

For methods for clearing a term of negative exponents study the following:

Illustrative examples.

1. $2a^{-1} = 2 \cdot \frac{1}{a} = \frac{2}{a}$. 2. $a^2b^{-3}c = a^2 \cdot \frac{1}{b^3} \cdot c = \frac{a^2c}{b^3}$.

3. $$\frac{2a^{-1}b}{x^{-2}y} = \frac{2 \cdot \frac{1}{a} \cdot b}{\frac{1}{x^2}y} = \frac{\frac{2b}{a}}{\frac{y}{x^2}} = \frac{2bx^2}{ay}.$$

Example 3 makes evident the following rule:

A factor may be moved from numerator to denominator, or from denominator to numerator, without changing the value of the fraction if the sign of the exponent of the factor is changed.

Negative exponents furnish a convenient method for expressing small decimals. For instance $.00005=\frac{5}{100000}=5\cdot10^{-5}$.

Exercise 163

Clear each of the following of negative exponents and simplify if possible:

1. $2a^{-3}bc^{-1}$. Ans. $\frac{2b}{a^3c}$.
2. $3^{-1}ab^{-1}$.
3. $a^{-1}bc^{-1}d$.
4. $\frac{2a^{-1}}{bc^{-1}}$.
5. $\frac{3^{-1}ab^{-1}c}{2a^{-1}b^2c^{-2}}$.
6. $\frac{m(a-b)^{-1}}{x}$.
7. $\frac{2(a-b)^{-1}}{3^{-1}(c-d)^{-2}}$. Ans. $\frac{6(c-d)^2}{a-b}$.
8. $\frac{3a^{-1}(x-y)^{-2}c}{2^{-2}a^{-3}(x-y)c^{-2}}$
9. $\frac{a^{-1}-b^{-1}}{a^{-1}+b^{-1}}=\frac{\frac{1}{a}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b}}=\frac{b-a}{b+a}$.
10. $\frac{x^{-1}+y^{-1}}{x^{-1}-y^{-1}}$.
11. $\frac{x^{-1}-y^{-1}}{x^{-2}-y^{-2}}$.
12. $\frac{x^{-2}-y^{-2}}{x^{-3}-y^{-3}}$.
13. $\frac{x^{-1}-2y^{-1}}{x^{-2}-4y^{-2}}=\frac{\frac{1}{x}-\frac{2}{y}}{\frac{1}{x^2}-\frac{4}{y^2}}=?$
14. $\frac{a^{-1}-3b^{-1}}{a^{-3}-27b^{-3}}$.
15. $\frac{4^{-1}-3^{-1}}{4^{-1}+3^{-1}}$. Ans. $-\frac{1}{7}$.
16. $\frac{3^{-1}-5^{-1}}{3^{-3}-5^{-3}}$.
17. $2^{-2}\cdot3\cdot4\cdot6^{-2}$.
18. $\frac{2^a\cdot2^{-3}-2^{-2}}{2^a\cdot2^{-2}-2^{-1}}$.
19. $5^{-2}\cdot25\cdot2^{-3}\left(\frac{1}{4}\right)^{-2}\cdot2$.

Write without denominators:

20. $\frac{2a}{bc}=2ab^{-1}c^{-1}$.

21. $\frac{3xy}{2m^2}$.

22. $\frac{3a}{a-b}=3a(a-b)^{-1}$.

23. $\frac{3xy}{x-y}$.

24. $\frac{2bc}{b(b-c)^2}$.

25. $\frac{3}{x^2-2xy+y^2}$.

26. $\frac{a^2-2ab+b^2}{m^2+mn+n^2}$.

Exercise 164

First perform the indicated operation, then clear the result of negative exponents.

1. $(x^{-1}-y^{-1})^2$. 2. $(x^{-2}-2x^{-1}y^{-1}+y^{-2})\ (x^{-1}-y^{-1})$.
3. $(a^{-1}-b^{-1})^3$. 4. $(x^{-2}-x^{-1}-6)\ (x^{-1}-2)$.
5. $(x^{-2}-4)\div(x^{-1}-2)$. 6. $(x^{-3}-y^{-3})\div(x^{-1}-y^{-1})$.
7. $(x^{-3}+y^{-3})\div(x^{-1}+y^{-1})$.
8. $(a^{-5}+b^{-5})\div(a^{-1}+b^{-1})$.
9. $(x^{-3}-3x^{-2}+3x^{-1}-1)\div(x^{-1}-1)$.
10. $(x^{-4}-x^{-2}+2x^{-1}-1)\div(x^{-2}-x^{-1}+1)$.
11. $(x^{-2}-2x^{-1}+1)^2$. 12. $(x^{-1}y^2-x^2y^{-1})^3$.
13. $(2a^{-1}-7-3a)\ (4a^{-1}+5)$.
14. $(a^{-1}b+2+ab^{-1})\ (a^{-1}b-2+ab^{-1})$.

170. Meaning of the fractional exponent.

Since $x^{\frac{1}{2}}\cdot x^{\frac{1}{2}}=x^{\frac{1}{2}+\frac{1}{2}}=x^{\frac{2}{2}}$ or x (Typeform I), therefore $x^{\frac{1}{2}}$ is one of the two equal factors of x, or $x^{\frac{1}{2}}$ is the square root of x, i.e. $x^{\frac{1}{2}}=\sqrt{x}$. Similarly $x^{\frac{1}{3}}\cdot x^{\frac{1}{3}}\cdot x^{\frac{1}{3}}=x^{\frac{3}{3}}$ or x. $\therefore x^{\frac{1}{3}}=\sqrt[3]{x}$. Also $x^{\frac{2}{3}}\cdot x^{\frac{2}{3}}\cdot x^{\frac{2}{3}}=x^{\frac{6}{3}}$ or x^2. $\therefore x^{\frac{2}{3}}=\sqrt[3]{x^2}$. But $x^{\frac{2}{3}}=(x^{\frac{1}{3}})^2$, $\therefore x^{\frac{2}{3}}=(\sqrt[3]{x})^2$ and $\sqrt[3]{x^2}=(\sqrt[3]{x})^2$.

Hence the **principle**:

The denominator of a fractional exponent indicates what root of the number is to be found and the numerator indicates what power.

35. $27^{-\frac{1}{3}} \cdot 8^{-\frac{1}{3}} \div 4^{\frac{1}{2}} \cdot 6^2$.

36. $12^2 \div 3^3 \div 4 = (2^2 \cdot 3)^2 \div 3^3 \div 2^2 = 2^4 \cdot 3^2 \div 3^3 \div 2^2 = 2^2 \cdot 3^{-1} = \frac{4}{3}$.

37. $(18)^2 \div 6^2 \cdot 8^{\frac{2}{3}} \div 24$. **38.** $6^{-\frac{1}{2}} \cdot 4^2 \cdot 12^{\frac{1}{2}} \div 18^{-\frac{1}{2}} \div 6^{-\frac{1}{2}} \cdot 2^{\frac{1}{2}}$.

173. Simplification of radicals.

A **radical** is a root of a number indicated by a radical sign. If the required root can be found, the radical is said to be a **rational number;** if it cannot be found it is an **irrational number** or a **surd.** $\sqrt{4}$, $\sqrt[3]{27}$, $\sqrt{a^2b^4}$, . . . are rational numbers. $\sqrt{5}$, $\sqrt[3]{10}$, $\sqrt[4]{a^3b^2}$, . . . are irrational numbers or surds. (See § 111.)

The number over the radical sign indicates the **order** of the radical. When no number is written, the square root is indicated, and the radical is of the second order; $\sqrt{2}$ is of the second order, $\sqrt[3]{2}$ is of the third order, and $\sqrt[n]{2}$ is of the nth order.

The radicals treated in Chapter VII were nearly all of the second order and were simplified, evaluated, added, subtracted, multiplied, and divided. Fractional exponents will be found helpful in explaining these processes and in dealing with higher orders of radicals.

Radicals may be simplified in several ways.

I. By reducing the order.

II. By removing a rational factor.

III. By removing the denominator if the radical is a fraction.

Illustrative examples.

1. $\sqrt[4]{9} = \sqrt[4]{3^2} = 3^{\frac{2}{4}} = 3^{\frac{1}{2}} = \sqrt{3}$. 3. $\sqrt{50} = \sqrt{2 \cdot 25} = \sqrt{25}\,\sqrt{2} = 5\sqrt{2}$.

2. $\sqrt[6]{125} = \sqrt[6]{5^3} = 5^{\frac{3}{6}} = 5^{\frac{1}{2}} = \sqrt{5}$. 4. $\sqrt[3]{40} = \sqrt[3]{5 \cdot 8} = \sqrt[3]{8}\,\sqrt[3]{5} = 2\sqrt[3]{5}$.

5. $\sqrt{\frac{2}{5}} = \sqrt{\frac{2 \cdot 5}{5 \cdot 5}} = \sqrt{\frac{10}{25}} = \sqrt{10 \cdot \frac{1}{25}} = \frac{1}{5}\sqrt{10}$.

6. $\sqrt[3]{\frac{3}{4a}} = \sqrt[3]{\frac{3 \cdot 2a^2}{4a \cdot 2a^2}} = \sqrt[3]{\frac{6a^2}{8a^3}} = \frac{1}{2a}\sqrt[3]{6a^2}$.

Illustrative examples 3, 4, 5, and 6 are worked under the following—

Rule. *In multiplication or division radicals of the same order may be placed under the same radical sign and their product or quotient taken; and, vice versa, a radical expression may be dissolved into radical factors of the same order.*

This law is apparent if the radicals are written as quantities with fractional exponents.

For $\sqrt[3]{a} \cdot \sqrt[3]{b} = (a^{\frac{1}{3}} \cdot b^{\frac{1}{3}}) = (ab)^{\frac{1}{3}} = \sqrt[3]{ab}$

and $\sqrt[3]{ab} = (ab)^{\frac{1}{3}} = a^{\frac{1}{3}} \cdot b^{\frac{1}{3}} = \sqrt[3]{a} \cdot \sqrt[3]{b}.$

Exercise 167

Follow the illustrative examples in simplifying the following:

1. $\sqrt[6]{36}.$ **2.** $\sqrt[4]{25}.$ **3.** $\sqrt[6]{216}.$ **4.** $\sqrt[6]{125}.$

5. $\sqrt{12}.$ **6.** $\sqrt[4]{48}.$ **7.** $\sqrt[3]{54}.$ **8.** $\sqrt[3]{250}.$

9. $\sqrt[4]{16a^5}.$ **10.** $\sqrt[3]{81a^4b^2}.$ **11.** $\sqrt[3]{56x^3y^4}.$ **12.** $\sqrt[3]{686}.$

13. $\sqrt[4]{128}.$ **14.** $\sqrt[3]{1024}.$ **15.** $\sqrt[5]{729}.$ **17.** $\sqrt[6]{128}.$

18. $\sqrt[3]{144}.$ **19.** $\sqrt{\frac{2}{5}}.$ **20.** $\sqrt{\frac{a}{b}}.$ **21.** $\sqrt[3]{\frac{a}{b}}.$

Ans. $\frac{1}{b}\sqrt[3]{ab^2}.$ **22.** $\sqrt[3]{\frac{16}{9a}}.$ **23.** $\sqrt[3]{\frac{1}{2}}.$

24. $\sqrt[4]{\frac{1}{2}}.$ **25.** $\sqrt[5]{\frac{1}{2}}.$ **26.** $\sqrt[3]{\frac{2}{25a}}.$

27. $2\sqrt[3]{\frac{54}{4a^2}}.$ **28.** $ab\sqrt[4]{\frac{2}{a^3b^2}}.$ **29.** $\frac{4}{3}\sqrt[3]{\frac{x^4}{16y^2}}.$

30. $\sqrt{\frac{1}{a+b}}.$ **31.** $\sqrt[3]{\frac{a}{(a+b)^2}}.$ **32.** $\sqrt[3]{\frac{8a^2}{m+n}}.$

33. $(x+y)\sqrt{\frac{x-y}{x+y}}.$ **34.** $(a+b)\sqrt[3]{\frac{a-b}{(a+b)^2}}.$

35. $4\pi\sqrt{\frac{3}{2\pi}} = 2\sqrt{6\pi}.$ **36.** $\frac{s}{3}\sqrt{\frac{s}{4\pi}}.$ **37.** $4\pi\sqrt[3]{\frac{9v^2}{16\pi^2}}.$

Exercise 168

First simplify and then evaluate correct to .001 *each of the following expressions, using the tables following* § 238 *for all square and cube roots.*

1. $\sqrt{\frac{2}{3}}=\frac{1}{3}\sqrt{6}=\frac{1}{3}(2.446)=?$
2. $\sqrt[3]{\frac{2}{5}}$. 3. $\sqrt{\frac{4}{5}}$. 4. $\sqrt[3]{\frac{3}{4}}$. 5. $\sqrt[3]{\frac{2}{7}}$.
6. $2-\sqrt{3}$. 7. $4-\sqrt{5}$. 8. $\sqrt[3]{\frac{1}{12}}$.
9. $\sqrt{512}$. 10. $\sqrt[3]{432}$. 11. $\sqrt{128}$.
12. $\sqrt{320}$. 13. $\sqrt{700}$. 14. $\sqrt[3]{500}$.
15. $\sqrt[3]{2000}$. 16. $\sqrt{3}-\sqrt{2}$.

174. Addition and subtraction of radicals.

Since every radical may be written without a radical sign by the use of fractional exponents, evidently the same rules will apply in the addition or subtraction of radicals that apply to other algebraic forms. We know that $2a^2+3a^2=5a^2$ and that $3a^{\frac{1}{2}}+5a^{\frac{1}{2}}=8a^{\frac{1}{2}}$. $\therefore 3\sqrt{a}+5\sqrt{a}=8\sqrt{a}$.

Exercise 169

Simplify and collect all similar terms:

1. $\sqrt{24}-\sqrt{6}+\sqrt{150}=2\sqrt{6}-\sqrt{6}+5\sqrt{6}=6\sqrt{6}$.
2. $\sqrt{32}-\sqrt{18}-\sqrt{8}+\sqrt{50}$. 3. $\sqrt{27}-\sqrt{12}+\sqrt{75}$.
4. $\sqrt[3]{16}-\sqrt[3]{250}+\sqrt[3]{54}$. Ans. 0.
5. $\sqrt{54}+2\sqrt{24}-\sqrt{96}$.
6. $2\sqrt[3]{81}-\sqrt[3]{192}+3\sqrt[3]{24}$.
7. $4\sqrt{\frac{3}{4}}-8\sqrt{\frac{3}{16}}+\sqrt{24}$.
8. $5\sqrt{3}+2\sqrt{48}-5\sqrt{108}$.
9. $\sqrt[3]{48}+\sqrt[3]{162}-\sqrt[3]{384}$.
10. $\sqrt[3]{500}+\sqrt[3]{256}-\sqrt[3]{32}-\sqrt[3]{108}$.
11. $\sqrt[3]{40}+5\sqrt[3]{\frac{1}{25}}-4\sqrt[3]{320}-\sqrt[3]{\frac{5}{8}}-\sqrt[3]{135}$.
12. $2\sqrt{4a^3}+\sqrt{9a^3}+\sqrt{25c^3}-\sqrt{36a^3}$.

13. $2\sqrt{\dfrac{x}{y}} - \sqrt{x^3y} - 2\sqrt{\dfrac{y}{x}} = \left(\dfrac{2}{y} - x - \dfrac{2}{x}\right)\sqrt{xy}$

or $\dfrac{(2x-x^2y-2y)}{xy}\sqrt{xy}.$

14. $\sqrt{\dfrac{a}{b}} + \sqrt{\dfrac{b}{a}} - 2\sqrt{ab}.$

15. $a\sqrt{(a^2-b^2)(a+b)} + 2ab\sqrt{a-b} - b\sqrt{(a-b)^3}.$

175. Comparison of values.

The relative values of radicals of different orders may be determined by the aid of fractional exponents.

Illustrative examples.

1. Which is the greater, $\sqrt{3}$ or $\sqrt[3]{5}$?

Now $\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{3}{6}} = \sqrt[6]{3^3} = \sqrt[6]{27}$,

and $\sqrt[3]{5} = 5^{\frac{1}{3}} = 5^{\frac{2}{6}} = \sqrt[6]{5^2} = \sqrt[6]{25}$.

$\therefore$ $\sqrt{3}$ is greater than $\sqrt[3]{5}$.

2. Arrange in ascending order of magnitude $\sqrt[3]{3}$, $\sqrt[4]{5}$, and $\sqrt[6]{10}$.

Reducing to the 12th order (the L. C. M. of 3, 4, and 6),

$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{4}{12}} = \sqrt[12]{3^4} = \sqrt[12]{81}$,

$\sqrt[4]{5} = 5^{\frac{1}{4}} = 5^{\frac{3}{12}} = \sqrt[12]{5^3} = \sqrt[12]{125}$,

and $\sqrt[6]{10} = 10^{\frac{1}{6}} = 10^{\frac{2}{12}} = \sqrt[12]{10^2} = \sqrt[12]{100}$.

$\therefore$ $\sqrt[3]{3} < \sqrt[6]{10} < \sqrt[4]{5}$.

Let the student state the process in the form of a rule.

Exercise 170

1. Which is the greater, $\sqrt[3]{4}$ or $\sqrt[4]{6}$?
2. Compare $\sqrt{2}$ and $\sqrt[5]{6}$.
3. Compare $\sqrt{7}$ and $\sqrt[3]{18}$.
4. Compare $\sqrt[4]{7}$ and $\sqrt[6]{19}$.

Arrange in ascending order:

5. $\sqrt{7}$, $\sqrt[3]{20}$, and $\sqrt[4]{50}$.
6. $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{5}$, and $\sqrt[6]{10}$.
7. $2\sqrt{3}$, $3\sqrt[3]{2}$, and $\sqrt[6]{2880}$. **Hint.** $2\sqrt{3} = \sqrt{12}$.
8. $2\sqrt{3}$, $2\sqrt[3]{5}$, and $\sqrt[6]{1700}$.
9. $5\sqrt{2}$, $2\sqrt[3]{6}$, and $4\sqrt[4]{10}$.

176. Multiplication and division of radicals.

Radicals of the same order may be placed under the common index and their product or quotient taken. (See § 173.)

For $\sqrt[3]{3} \cdot \sqrt[3]{5} = 3^{\frac{1}{3}} \cdot 5^{\frac{1}{3}} = (3 \cdot 5)^{\frac{1}{3}} = \sqrt[3]{3 \cdot 5} = \sqrt[3]{15}$

and $\sqrt[3]{12} \div \sqrt[3]{3} = (12)^{\frac{1}{3}} \div 3^{\frac{1}{3}} = (12 \div 3)^{\frac{1}{3}} = \sqrt[3]{12 \div 3} = \sqrt[3]{4}.$

By the use of fractional exponents, radicals of different orders may be transformed into equivalent radicals of a common order following the plan of the last paragraph and their product or quotient taken.

Illustrative examples.

1. $\sqrt{x} \cdot \sqrt[3]{x} = x^{\frac{1}{2}} \cdot x^{\frac{1}{3}} = x^{\frac{5}{6}} = \sqrt[6]{x^5}.$
2. $\sqrt[3]{5} \cdot \sqrt{2} = 5^{\frac{1}{3}} \cdot 2^{\frac{1}{2}} = 5^{\frac{2}{6}} \cdot 2^{\frac{3}{6}} = \sqrt[6]{5^2 \cdot 2^3} = \sqrt[6]{200}.$
3. $\sqrt[3]{a^2} \div \sqrt{a} = a^{\frac{2}{3}} \div a^{\frac{1}{2}} = a^{\frac{1}{6}} = \sqrt[6]{a}.$
4. $\sqrt[3]{6} \div \sqrt[4]{3} = 6^{\frac{1}{3}} \div 3^{\frac{1}{4}} = 6^{\frac{4}{12}} \div 3^{\frac{3}{12}}$
   $= \sqrt[12]{6^4 \div 3^3} = \sqrt[12]{1296 \div 27} = \sqrt[12]{48}.$

Exercise 171

Perform the indicated operations in each of the following:

1. $\sqrt{a} \cdot \sqrt[3]{a^2}$. Ans. $\sqrt[6]{a^7}$, or $a\sqrt[6]{a}$.
2. $\sqrt{x} \cdot \sqrt[5]{x^3}$.
3. $\sqrt[3]{a^2} \div \sqrt[4]{a}$.
4. $3\sqrt[3]{a^2} \div \sqrt{a}$.
5. $2\sqrt[4]{a^3} \cdot \sqrt{a}$.
6. $2\sqrt[4]{a^3} \div \sqrt{a}$.
7. $(3\sqrt{a}\sqrt[3]{b})^2$.
8. $(7\sqrt[3]{2x}\sqrt{y})^7$.
9. $(\sqrt{x} - \sqrt{y})^2$.
10. $(\sqrt{x} + \sqrt{y})^3$.
11. $(\sqrt{x} - \sqrt[3]{y})^2$.
12. $(\sqrt{x} - \sqrt[3]{y})^3$.
13. $(2a^{\frac{1}{3}} - 3b^{\frac{1}{2}})^2$.
14. $(a^{\frac{1}{3}} - b^{\frac{1}{2}})^3$.
15. $(x - \sqrt{xy} + y)(\sqrt{x} + \sqrt{y})$.
16. $(\sqrt{x^3} - \sqrt{y^3})(\sqrt{x^3} + \sqrt{y^3})$.
17. $(x - y) \div (\sqrt{x} - \sqrt{y})$.
18. $(x + y) \div (\sqrt[3]{x} + \sqrt[3]{y})$.
19. $(x - y) \div (\sqrt[4]{x} - \sqrt[4]{y}) \div (\sqrt[4]{x} + \sqrt[4]{y})$.

177. Rationalizing factor. It is convenient to transform a fraction with a radical denominator into an equivalent fraction with a rational denominator. This is known as rationalizing the denominator and depends upon the finding of a **rationalizing multiplier or factor.** (See § 120.)

Exercise 172

What is the simplest rationalizing factor for each of the following? Check by actual multiplication.

1. $\sqrt{7}$. Ans. $\sqrt{7}$.
2. $2\sqrt{5}$. Ans. $\sqrt{5}$.
3. $\sqrt[3]{4}$. Ans. $\sqrt[3]{2}$.
4. $\sqrt[3]{5}$.
5. $\sqrt[4]{8}$.
6. $\sqrt[4]{2}$.
7. $\sqrt[5]{a^4}$.
8. $\sqrt[3]{a^2b}$.
9. $\sqrt[3]{2ab^2}$.
10. $\sqrt[4]{27a^2b}$.
11. $2-\sqrt{3}$. Ans. $2+\sqrt{3}$.
12. $\sqrt{7}-\sqrt{5}$.
13. $2\sqrt{3}-3\sqrt{2}$.
14. $\sqrt{a}+\sqrt{b}$.
15. $\sqrt{5}-\sqrt{3}-\sqrt{2}$. Try $(\sqrt{5}+\sqrt{3}+\sqrt{2})\cdot\sqrt{6}$. Why?
16. $\sqrt{3}-\sqrt{2}-1$.
17. $\sqrt{a}-\sqrt{b}+c$.

When rationalizing the denominator of a fraction, multiply both terms by the simplest rationalizing factor of the denominator.

Illustrative examples.

1. $\dfrac{1}{\sqrt{2}}=\dfrac{1\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}=\dfrac{\sqrt{2}}{2}$.

2. $\dfrac{2}{\sqrt[3]{4}}=\dfrac{2\cdot\sqrt[3]{2}}{\sqrt[3]{4}\cdot\sqrt[3]{2}}=\dfrac{2\sqrt[3]{2}}{2}=\sqrt[3]{2}$.

3. $\dfrac{3}{\sqrt{3}-\sqrt{2}}=\dfrac{3(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}=\dfrac{3(\sqrt{3}+\sqrt{2})}{3-2}=3\sqrt{3}+3\sqrt{2}$.

Exercise 173

Transform into equivalent fractions with a rational denominator and simplify:

1. $\dfrac{2}{3\sqrt{2}}$. Ans. $\dfrac{\sqrt{2}}{3}$.
2. $\dfrac{2}{\sqrt{2}}$.
3. $\dfrac{2}{\sqrt[3]{2}}$.
4. $\dfrac{3}{\sqrt[3]{9}}$.
5. $\dfrac{a^2}{\sqrt{a}}$.
6. $\dfrac{2a}{\sqrt[3]{4a}}$.
7. $\dfrac{2ab}{\sqrt[4]{8a^2b^3}}$.

8. $\dfrac{12}{\sqrt[3]{6}}$. 9. $\dfrac{7}{2\sqrt[3]{49}}$. 10. $\dfrac{10}{\sqrt{50}}$. 11. $\dfrac{15}{\sqrt[3]{40}}$.

12. $\dfrac{3}{\sqrt{2}-1}$. 13. $\dfrac{4}{\sqrt{3}-1}$. 14. $\dfrac{5}{\sqrt{7}+\sqrt{2}}$.

15. $\dfrac{\sqrt{2}-4}{3\sqrt{2}-2\sqrt{3}}$. 16. $\dfrac{2\sqrt{3}-3\sqrt{2}}{2\sqrt{2}-3\sqrt{3}}$.

17. $\dfrac{a-b}{\sqrt{a}-\sqrt{b}}$. 18. $\dfrac{a\sqrt{b}-b\sqrt{a}}{b\sqrt{b}-a\sqrt{a}}$.

19. $\dfrac{28}{2\sqrt{6}-2\sqrt{2}}$. 20. $\dfrac{5}{\sqrt{14}-3}$. 21. $\dfrac{2\sqrt{3}-3\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.

22. $\dfrac{6}{\sqrt{3}+\sqrt{5}+\sqrt{2}}$. 23. $\dfrac{4}{\sqrt{3}+\sqrt{2}+1}$.

24. $\dfrac{ab}{\sqrt{a}+\sqrt{b}+\sqrt{a+b}}$.

178. Exponential equations.

Exercise 174

Solve each of the following for x:

1. $4^x=16$. (Since $16=4^2$, $\therefore 4^x=4^2$ and $x=2$.)

2. $3^x=27$. 3. $9^x=27$. $[(3^2)^x=3^3$ or $3^{2x}=3^3$. $\therefore 2x=3$ and $x=\frac{3}{2}$.$]$

4. $2^x=8$. 5. $4^x=8$. 6. $16^x=8$.

7. $27^x=9$. 8. $81^x=9$. 9. $25^x=\frac{1}{5}$.

10. $32^x=16$. 11. $8^x=\frac{1}{16}$. 12. $8^{x-2}=\frac{1}{16}$.

13. $(\frac{1}{3})^x=27$. 14. $3^{x-1}=27$. 15. $(\frac{1}{2})^{-x}=8$.

16. $x^{\frac{2}{3}}=4$. **Hint.** $(x^{\frac{1}{3}})^2=2^2 \therefore x^{\frac{1}{3}}=2$ and $x=8$.

17. $(\frac{1}{4})^x=16$. 18. $(\frac{1}{27})^x=81$.

19. $x^{\frac{1}{2}}=2$. 20. $x^{\frac{1}{3}}=9$. 21. $x^{\frac{1}{3}}=\frac{1}{4}$.

22. $x^{-\frac{2}{3}}=25$. Ans. $x=\frac{1}{125}$.

23. $x^{-\frac{1}{2}}=\frac{9}{4}$. 24. $x^{-\frac{3}{2}}=1000$.

25. $4a^{-\frac{5}{3}}=\frac{1}{8}$. **Suggestion.** $a^{-\frac{5}{3}}=\dfrac{1}{32}$.

26. $\frac{8}{5}a^{-\frac{2}{3}}=\frac{5}{8}$. 27. $\frac{27}{8}a^{-3}=125$.

179. A **binomial quadratic surd** is a binomial one or both of whose terms is a surd of the second degree. $2+\sqrt{3}$, $\sqrt{2}-3\sqrt{5}$, and $\sqrt{5}-w$ are binomial quadratic surds. The general or typeform is $a+b\sqrt{c}$.

It is possible to obtain the square root of certain binomial quadratic surds, as follows:

By multiplication, the value of $(\sqrt{2}+\sqrt{3})^2$ is found to be $5+2\sqrt{6}$. The value of $(\sqrt{a}+\sqrt{b})^2$ is $a+b+2\sqrt{ab}$ where the rational part, $a+b$, is the sum of the two radicands and the radical part $\sqrt{ab}$ has for its radicand the product of the two radicands of the given binomial.

Therefore $\sqrt{7-2\sqrt{10}}$ can be simplified if 7 is the sum of two factors of 10. These factors are evidently 5 and 2 and $\sqrt{7-2\sqrt{10}}=\sqrt{5}-\sqrt{2}$ or $\sqrt{2}-\sqrt{5}$.

Check by finding the value of $(\sqrt{5}-\sqrt{2})^2$ and also $(\sqrt{2}-\sqrt{5})^2$.

The process may be stated as follows:

To find the square root of a binomial quadratic surd, transform the surd term into the form $2\sqrt{m}$. If m has two factors, a and b, whose sum is the rational term of the binomial, the square root of the binomial is $\sqrt{a}+\sqrt{b}$ or $\sqrt{a}-\sqrt{b}$ according to the sign of the surd term in the given binomial.

Illustrative examples.

1. $\sqrt{7+4\sqrt{3}}=\sqrt{7+2\sqrt{12}}=\sqrt{4}+\sqrt{3}$ or $2+\sqrt{3}$.
   Check by squaring $2+\sqrt{3}$.

2. $\sqrt{9-4\sqrt{5}}=\sqrt{9-2\sqrt{20}}=\sqrt{5}-\sqrt{4}$ or $\sqrt{5}-2$.

(**Note.** $\sqrt{9-4\sqrt{5}}$ may be written $2-\sqrt{5}$ but $\sqrt{5}-2$ is the positive root and $2-\sqrt{5}$ is the negative root. Why?)

The root, $\sqrt{5}-2$, is the principal root.
For discussion of principal root, see § 171.

Exercise 175

Find the positive square root of each of the following:

1. $8-2\sqrt{15}$. **2.** $8-2\sqrt{7}$. **3.** $12-6\sqrt{3}$.

4. $11-2\sqrt{30}$. **5.** $11-4\sqrt{6}$. **6.** $9+2\sqrt{18}$.

7. $31-10\sqrt{6}$. **8.** $a+b-2\sqrt{ab}$.

9. $2a+3b+2\sqrt{6ab}$. **10.** $\frac{3}{2}-2\sqrt{\frac{1}{2}}$.

11. $9\frac{1}{3}-2\sqrt{3}$. **12.** $a^2+b+2a\sqrt{b}$.

180. Imaginary numbers and the imaginary unit.

The square root of a number has been defined as one of its two equal factors. Since a negative number cannot have two equal factors, its square root cannot be found.

The product of $(+2)(+2)$ and also of $(-2)(-2)$ is $+4$, therefore the square root of $+4$ (written $\pm\sqrt{4}$) may be either $+2$ or -2. Similarly, $(+\sqrt{3})(+\sqrt{3})$ and $(-\sqrt{3})(-\sqrt{3})$ both equal 3, therefore the square root of 3 may be either $+\sqrt{3}$ or $-\sqrt{3}$.

But -4 is not the product of two equal factors therefore its square root can only be expressed, as $\sqrt{-4}$. Similarly for the square root of -3 and of all other negative numbers. The fourth root of $+16$ is either $+2$ or -2 (prove by multiplication), but there are no four equal factors whose product is -16 and the fourth root of -16 can only be expressed, as $\sqrt[4]{-16}$.

Such indicated even roots of negative numbers are known as **imaginary numbers**, or simply **imaginaries.** So far in our study of mathematics, all numbers have been real numbers.

Such numbers as $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-7}$, and $\sqrt{-30}$ may be written as the product of two factors, that is $\sqrt{-2}=\sqrt{2}\cdot\sqrt{-1}$, $\sqrt{-3}=\sqrt{3}\cdot\sqrt{-1}$, $\sqrt{7}=\sqrt{7}\cdot\sqrt{-1}$, and $\sqrt{-30}=\sqrt{30}\cdot\sqrt{-1}$. One of these factors, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{7}$, $\sqrt{30}$, . . . , is a real number and the other, $\sqrt{-1}$, is a new number which has been conveniently called the **imaginary unit.** Every indicated

square root of a negative number may be separated into two factors, one of which is the imaginary unit.

The study of imaginary numbers is largely the interpretation and application of this imaginary unit.

The letter i is frequently used as the symbol for the imaginary unit. $3\sqrt{-1}$ is written $3i$, $\sqrt{-2}$ is written $i\sqrt{2}$, and $\sqrt{-3}$ is written $i\sqrt{3}$.

Now by the definition of a square root we know that $(\sqrt{-1})^2=-1$, or $i^2=-1$. Similarly,
$(\sqrt{-1})^3=(\sqrt{-1})^2\cdot(\sqrt{-1})=-\sqrt{-1}$, or $i^3=i^2\cdot i=-i$.

Also $(\sqrt{-1})^4=[(\sqrt{-1})^2]^2=(-1)^2=+1$,
or $i^4=(i^2)^2=(-1)^2=+1$.

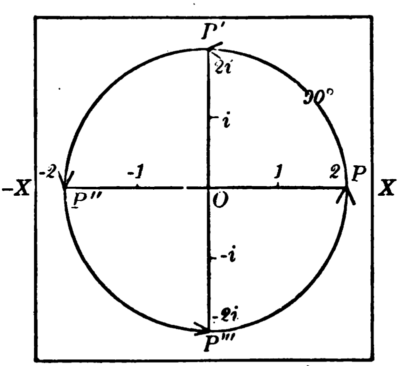
Or, in tabular form:

$$\begin{aligned}
\sqrt{-1} &= i\\
(\sqrt{-1})^2 &= i^2=-1\\
(\sqrt{-1})^3 &= i^3=-\sqrt{-1}, \text{ or } -i\\
(\sqrt{-1})^4 &= i^4=+1.
\end{aligned}$$

181. Interpretation of the imaginary unit.

It is from an application of this table to the number scale that mathematicians have obtained a reasonable and very useful interpretation of the meaning of the imaginary unit.

On the number scale XX′, let P be taken at the point $+2$, that is, let the line-segment OP be $+2$ units long. Similarly, let P″ be taken at the point -2 and the line-segment OP″ be -2 units long.



The line-segment OP may be made to coincide with the line-segment OP″ by turning it on the plane about O as a pivot through an angle of 180°. But the

number, $+2$, may be converted into the number -2 by multiplying $+2$ twice by $\sqrt{-1}$, or by i^2.

If the angle of 180° is divided into two equal angles each 90°, then, since multiplication by $(\sqrt{-1})^2$ is equivalent to turning the segment through an angle of 180°, multiplication by $\sqrt{-1}$ may be thought of as geometrically equivalent to turning a segment counter-clockwise through an angle of 90°. And we may define the imaginary unit as an operator that rotates a segment about a point on a plane through an angle of 90° with each application. This interpretation of the meaning of the imaginary unit expands our idea of number from that of position on a directed line to position on a plane. Its application will be found very helpful in many places in higher mathematics.

The sum or the difference of a real number and an imaginary number is called a **complex number,** such as $2+\sqrt{-3}$ and $\sqrt{2}+\sqrt{-5}$.

Its general type may be written $a+b\sqrt{-1}$, or $a+bi$, where a and b are real numbers. (For interpretation, see Chap. XXII.)

182. Numbers that have $\sqrt{-1}$, or i, as a factor may be treated as real numbers have been treated provided care is used in simplifying all powers of $\sqrt{-1}$, or i, above the first under the typeform $(\sqrt{-1})^2 \equiv -1$.

Illustrative examples.

1. $\sqrt{-12}+\sqrt{-27}-3\sqrt{-\frac{1}{3}}=\sqrt{12}\ \sqrt{-1}+\sqrt{27}\ \sqrt{-1}-3\sqrt{\frac{1}{3}}\ \sqrt{-1}$ $=2i\sqrt{3}+3i\sqrt{3}-i\sqrt{3}=4i\sqrt{3}.$

2. $(\sqrt{-18})\ (2\sqrt{-12})=(3i\sqrt{2})\ (4i\sqrt{3})=12i^2\sqrt{6}=-12\sqrt{6}.$

3. $(\sqrt{a}+\sqrt{-b})\ (\sqrt{a}-\sqrt{-b})=(\sqrt{a}+i\sqrt{b})\ (\sqrt{a}-i\sqrt{b})=a-i^2b=$ $a+b.$

4. $$\frac{3}{\sqrt{2}-\sqrt{-1}}=\frac{3(\sqrt{2}+i)}{(\sqrt{2}-i)\ (\sqrt{2}+i)}=\frac{3(\sqrt{2}+i)}{2-i^2}=\frac{3(\sqrt{2}+i)}{3}=\sqrt{2}+i.$$

Exercise 176

Follow the plan of the illustrative examples in simplifying the following:

1. $\sqrt{-16}+\sqrt{-9}-\sqrt{-25}$. Ans. $2i$.
2. $\sqrt{-27}+\sqrt{-75}-\sqrt{8}$. Ans. $8i\sqrt{3}-2\sqrt{2}$.
3. $2\sqrt{-27}-3\sqrt{-25}+4\sqrt{-9}$.
4. $5\sqrt{-81a^2}-8\sqrt{-36a^2}$.
5. $2\sqrt{-8}-4\sqrt{-12}+3\sqrt{-18}+5\sqrt{-27}$.
6. $\sqrt{-50}-\sqrt{-75}+\sqrt{-72}+\sqrt{-98}$.
7. $4\sqrt{-96}+2\sqrt{-54}+\sqrt{-150}-\sqrt{-125}$.
8. $2\sqrt{-a^3}+\sqrt{-4a}+\sqrt{-9a^5}$.
9. $3\sqrt{-x^3}-\sqrt{-49a^2x}+\sqrt{-36a^4x^5}$.
10. $(3\sqrt{-1})(\sqrt{-9})$. **11.** $(2\sqrt{-3})(3\sqrt{-2})$.
12. $(\sqrt{-18})(\sqrt{-9})(\sqrt{-3})^2$. **13.** $(-\sqrt{-12})(\sqrt{-18})$.
14. $(\sqrt{-21a})(\sqrt{-14b})(\sqrt{-6c})$.
15. $(\sqrt{-6a^3})(\sqrt{-3a})(\sqrt{-2b^3})$.
16. $(a\sqrt{-1})(b\sqrt{-1})(c\sqrt{-1})(d\sqrt{-1})$.
17. $(3+\sqrt{-2})(3-\sqrt{-2})$.
18. $(2\sqrt{a}-3\sqrt{-b})(2\sqrt{a}+3\sqrt{-b})$.
19. $(\sqrt{5}+2\sqrt{-3})^2$. **20.** $(\sqrt{3}+\sqrt{-2})^3$.
21. $(\sqrt{3}+2\sqrt{-2})(2\sqrt{3}-\sqrt{-2})(10-3\sqrt{-6})$.
22. $(\sqrt{-2}+\sqrt{-3}-\sqrt{-5})^2$.
23. $(\sqrt{-a}+\sqrt{-b}+\sqrt{-c}-\sqrt{-9})^2$.
24. $\dfrac{6}{\sqrt{-3}}$. **25.** $\dfrac{2a}{\sqrt{-a}}$. **26.** $\dfrac{5}{\sqrt{3}-\sqrt{-2}}$.
27. $\dfrac{ab}{a\sqrt{-b}-b\sqrt{-a}}$. **28.** $\dfrac{6(\sqrt{5}+\sqrt{-7})}{\sqrt{5}-\sqrt{-7}}$.

Find one square root for each of the following:

29. $1+2\sqrt{-6}$. Ans. $\sqrt{3}+\sqrt{-2}$.
30. $2-2\sqrt{-15}$. Ans. $\sqrt{5}-\sqrt{-3}$.

31. $4-6\sqrt{-5}$. **32.** $1-12\sqrt{-2}$.

33. If $x=-1+\sqrt{-1}$, find the value of x^2+2x+7. Ans. 5.

34. If $x=\dfrac{3-\sqrt{-3}}{2}$, find the value of x^2-3x+3.

35. Does $x=\dfrac{-1+\sqrt{-7}}{4}$ satisfy $2x^2+x+1=0$?

Exercise 177. Review

Find one square root for each of the following polynomials (See § 109):

1. $x^4-2x^3-3x^2+4x+4$.

2. $4a^4-12a^3b+a^2b^2+12ab^3+4b^4$.

3. $4a^2+12ab+9b^2-20ac-30bc+25c^2$.

4. $n^4-2n^3+2n^2-n+\frac{1}{4}$.

5. $\dfrac{9a^4}{16}-\dfrac{3a^3}{4}+\dfrac{5a^2}{8}-\dfrac{a}{4}+\dfrac{1}{16}$.

6. $9a^{\frac{4}{3}}-12a^{\frac{3}{3}}-8a^{\frac{2}{3}}+8a^{\frac{1}{3}}+4$.

7. $9x^{-\frac{4}{3}}-12x^{-1}-14x^{-\frac{2}{3}}+12x^{-\frac{1}{3}}+9$.

8. $4n^{-2}-12n^{-1}+17-12n+4n^2$.

Find one square root for each of the following numbers (See § 110):

9. 50625. **10.** 139129. **11.** 91385.29.

12. 18.2329. **13.** 65124900. **14.** 2579236.

Find one square root for each of the following, correct to .001:

15. 39. **16.** 2. **17.** 7. **18.** 29.3.

19. 338.105. **20.** .89.

21. Multiply $2n^{-2}-3n^{-1}-2$ by $3n^{-2}+n^{-1}+4$.

22. Multiply $2a^{\frac{2}{3}}-3a^{\frac{1}{3}}+5$ by $a^{\frac{2}{3}}-2a^{\frac{1}{3}}-2$.

23. Divide $6x-7x^{\frac{4}{5}}+6x^{\frac{3}{5}}-12x^{\frac{2}{5}}+4x^{\frac{1}{5}}+3$ by $3x^{\frac{2}{5}}-2x^{\frac{1}{5}}-1$. Check with $x=32$.

Express the following in simplest radical form:

24. $2\sqrt{12}-\sqrt{75}+6\sqrt{\frac{1}{3}}$.

25. $\sqrt{20}-4\sqrt{\frac{1}{5}}-5^{-\frac{1}{2}}+5^{\frac{3}{2}}$.

26. $\sqrt{12}-9\sqrt{\tfrac{1}{3}}+8^{-\frac{1}{3}}\cdot 6^0.$

27. $\dfrac{3}{\sqrt{5}-\sqrt{2}}+\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}.$

28. $\sqrt{24}+\dfrac{3}{\sqrt{6}}-\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}.$

29. $(\tfrac{4}{5})^{\frac{1}{2}}+13\sqrt{\tfrac{1}{5}}+\sqrt[4]{\dfrac{25}{81}}.$

30. $\dfrac{\sqrt{-5}+\sqrt{-3}}{\sqrt{-5}-\sqrt{-3}}-\dfrac{\sqrt{-5}-\sqrt{-3}}{\sqrt{-5}+\sqrt{-3}}.$

Express the following with positive exponents:

31. $\dfrac{m^{-1}-n^{-1}}{m^{-2}-n^{-2}}\div\dfrac{m^{-1}n^{-1}}{(m+n)^{-1}}.$

32. $\dfrac{ab^{-1}+a^{-1}b+1}{a^{-3}-b^{-3}}\cdot\dfrac{a^{-2}-b^{-2}}{a^{-1}b^{-1}}.$

33. $2\cdot 8^{\frac{2}{3}}-\sqrt{2}(18)^{\frac{1}{2}}-2\cdot 5^0+\left(\dfrac{a+b}{ab}\right)^{-1}-\dfrac{1}{a^{-1}+b^{-1}}.$

Simplify the following and approximate the results, correct to .001:

34. $\sqrt{8}+5\sqrt{\tfrac{1}{2}}-18^{\frac{1}{2}}+20^{-1}\cdot 50^{\frac{3}{2}}.$

35. $\dfrac{6}{\sqrt{3}}-9\sqrt{\tfrac{1}{3}}+12^{\frac{1}{2}}-25^0.$

36. $8^{-\frac{2}{3}}+\dfrac{1}{16^{\frac{3}{4}}}-25^{\frac{1}{2}}+8^{-\frac{1}{3}}.$

Find one square root for each of the following:

37. $9-6\sqrt{2}.$

38. $2m+2\sqrt{m^2-n^2}.$

39. $2+2\sqrt{-15}.$

40. $a^2-b-2a\sqrt{-b}.$

41. If $x=-2-\sqrt{3}$, find the value of $x^2+4x+7.$

42. Does $x=\dfrac{3-\sqrt{-7}}{2}$ satisfy $x^2-3x+4=0$?

CHAPTER XV

LOGARITHMS

183. Introduction. Any positive number may be written as a power of some other positive number. For instance, $4=2^2$, $8=2^3$, $\frac{1}{9}=3^{-2}$. If a is greater than 0 and $a^x=b$, x is said to be the **logarithm** of b to the **base** a, and b is the **antilogarithm** of x to the base a. Written compactly the first statement is $\log_a b=x$, which is to be read "the logarithm of b to the base a is x."

Exercise 178

Find x by inspection in each of the following:

1. $\log_2 8=x$. **2.** $\log_3 27=x$. **3.** $\log_2 \frac{1}{8}=x$.

4. $\log_3 \frac{1}{9}=x$. **5.** $\log_4 8=x$. Ans. $x=\frac{3}{2}$.

6. $\log_{10} 100=x$. **7.** $\log_4 \frac{1}{8}=x$.

8. $\log_{10} .1=x$. Ans. $x=-1$. **9.** $\log_{10} .01=x$.

10. $\log_{10} .001=x$. **11.** $\log_9 27=x$.

12. $\log_8 4=x$. **13.** $\log_{\frac{1}{5}} 25=x$.

14. $\log_{16} 8=x$.

Since 6 is between 4 and 8, 6 is between the second and the third powers of 2, or it is more than 2^2, and less than 2^3. Written in the form used above we have $\log_2 6=2.\ldots$, or the logarithm of 6 to the base 2 is 2+ some decimal. When the value of this decimal is computed it is found to be approximately .5849. $\therefore 6=2^{2.5849}$, or $\log_2 6=2.5849$.

184. The integral part of the logarithm, or 2, is called the **characteristic**, and the decimal part, .5849, is called the **mantissa**. In like manner any positive number may be written as a power of 2 or of any other positive number except 1.

Common logarithms have 10 for their base and the accompanying tables give the mantissas of the logarithms to the base 10 of all the numbers from 100 to 999 approximately correct to the fourth place of decimals.

To find from the tables the logarithm of such a number as 457, we note first that 457 is more than 100 and less than 1000, therefore it is more than 10^2. For the decimal part, or the mantissa, we look in the tables down the first column at the left for the first two significant figures, 45, and then along their line across the page to the column headed 7, the third significant figure, where we find 6599. This is the approximate decimal value of the mantissa and $457 = 10^{2.6599}$, or $\log_{10} 457 = 2.6599$.

Since 10 is the most commonly used base for all logarithmic tables, it is not customary to name the base unless some other number is to be used, that is, $\log 457 = 2.6599$ will be the form and is to be read "logarithm of 457 is 2.6599."

Exercise 179

Find the logarithm for each of the following:

1. 247. (Write your answer $\log 247 = 2.3927$.)

2. 299. **3.** 425. **4.** 755. **5.** 333. **6.** 129. **7.** 999.

8. 500. **9.** 852. **10.** 102. **11.** 771. **12.** 101.

185. The characteristic.

Since $45.7 = \frac{1}{10}$ of 457, or $10^{-1}(457)$, therefore $45.7 = 10^{-1}\,10^{2.6599}$, or $10^{1.6599}$, that is, $\log 45.7 = 1.6599$.

Also $4.57 = \frac{1}{100}$ of 457, or $10^{-2}(457)$, therefore $\log 4.57 = 0.6599$.

Now $4570 = 10^1(457)$, therefore $\log 4570 = 3.6599$,
and $45700 = 10^2(457)$, therefore $\log 45700 = 4.6599$.

Evidently the characteristic depends upon the place of the decimal point in the number, the logarithm of which is to be determined.

The method for determining the characteristic of the logarithm of a number will appear from a study of the following table:

$1000 = 10^3$,	or $\log 1000 = 3$,		$\log 4570 = 3.6599$.
$100 = 10^2$,	or $\log 100 = 2$,		$\log 457 = 2.6599$.
$10 = 10^1$,	or $\log 10 = 1$,		$\log 45.7 = 1.6599$.
$1 = 10^0$,	or $\log 1 = 0$,		$\log 4.57 = 0.6599$.
$.1 = 10^{-1}$,	or $\log .1 = -1$,	or $\bar{1}$,	$\log .457 = \bar{1}.6599$.
$.01 = 10^{-2}$,	or $\log .01 = -2$,	or $\bar{2}$,	$\log .0457 = \bar{2}.6599$.
$.001 = 10^{-3}$,	or $\log .001 = -3$,	or $\bar{3}$,	$\log .00457 = \bar{3}.6599$.

It should be noticed that the mantissa is the same in the logarithms of 4570, 457, 45.7, etc., because 4570 is the same multiple of 1000 that 457 is of 100, 45.7 of 10, and 4.57 of 1.

The mantissa is always positive but the characteristic is negative when the number is less than unity.

Evidently the characteristic of the logarithm of a number with integral places is positive and one less than the number of integral places. The characteristic of the logarithm of a decimal number is negative and determined by counting the decimal point and the number of ciphers to the right of the decimal point and between it and the first significant figure, or it is determined by the decimal position of the first significant figure.

There are two ways of writing a negative characteristic: (1) as in the preceding table with a minus sign over the characteristic, or, (2) 10 is added to and subtracted from the logarithm, as follows:

$$\text{Log } .457 = \bar{1}.6599, \text{ or } 9.6599-10$$
$$\text{Log } .0457 = \bar{2}.6599, \text{ or } 8.6599-10$$
$$\text{Log } .00457 = \bar{3}.6599, \text{ or } 7.6599-10.$$

We will use the second way throughout this chapter.

Exercise 180

Name at sight the characteristic of each of the following:

1. 234. **2.** 23.4. **3.** 2.34. **4.** .234. **5.** .0234.
6. .000234. **7.** 999999. **8.** 77777.7. **9.** 100000000.
10. 1000.001. **11.** 10.000007. **12.** .000000001.

Find the mantissa from the tables and write the logarithm of each of the following:

13. .0999. Ans. log $.0999 = 8.9996 - 10$.
14. 57700. **15.** 4590. **16.** .00577. **17.** 3580000.
18. 93000000. **19.** 5280. **20.** .00004. **21.** .000000435.

186. To find an antilogarithm. If the logarithm of a number is known, it is only necessary to reverse the process in using the tables to locate the number.

What number has 2.5119 for its logarithm, or what is the antilogarithm of 2.5119?

A search through the tables of mantissas locates 5119 as the mantissa of 325 and, since the characteristic is 2, the antilogarithm of 2.5119 is 325.

Similarly, log $63300 = 4.8014$ and log $.0541 = 8.7332 - 10$.

Exercise 181

Find the antilogarithm of each of the following:

1. 1.2201. **2.** 5.3010. **3.** 3.4871. **4.** $8.8779 - 10$.
5. $7.6464 - 10$. **6.** $9.8779 - 10$. **7.** 0.3010. **8.** 7.9991.

187. The approximate logarithm of a number of four figures may be found from these tables in the following manner:

Given the number 2086. We find the mantissa of 208 which is 3181 and of 209 which is 3201. The difference between these two mantissas is 20. Now 2086 is $\frac{6}{10}$ of the interval between 2080 and 2090, therefore its logarithm should have the mantissa $3181 + 6/10$ of 20, or $3181 + 12$, making it 3193. Therefore log $2086 = 3.3193$.

The logarithm of 457.6 is 2.6605.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

Exercise 182

Find the approximate logarithm of each of the following:

1. 44.45. **2.** 234.7. **3.** 125800. **4.** .002755.
5. 3.141. **6.** 9990. **7.** 7926. **8.** 4.188. **9.** .3125.
10. .0007224. **11.** 6666. **12.** 1728. **13.** 1.732.

The fourth figure of an antilogarithm may be found approximately by a method similar to that used in the preceding paragraph.

Given that log $x = 3.3575$, find the number x. When we compare the mantissa 3575 with the table we find that it lies between 3560 and 3579 which are the mantissas of the logarithms of 227 and 228, respectively.

Evidently 3575 is the mantissa of a number between 227 and 228 and 3.3575 is the logarithm of a number between 2270 and 2280. Since 3579 is 19 more than 3560 and 3575 is 15 more than 3560, 3.3575 is the logarithm of a number greater than 2270 by 15/19 of 10 which is approximately 8. Therefore the antilogarithm of 3.3575 is 2278. Check this by finding log 2278.

188. To find the antilogarithm when the given mantissa is not in the table, find from the table the next lower mantissa with the corresponding three figures of the antilogarithm, subtract this mantissa of the table from the given mantissa, and divide this difference by the difference between this tabular mantissa and the next higher mantissa of the table. This quotient gives the fourth figure of the antilogarithm. The characteristic indicates the position of the decimal point.

Exercise 183

Find the antilogarithm of each of the following:

1. 2.5535. **2.** 3.6586. **3.** 1.9304. **4.** 0.4971.
5. 9.8239−10. **6.** 3.2375. **7.** 0.2386. **8.** 1.5000.
9. 6.8660−10. **10.** 2.9942. **11.** 9.0999−10.
12. 3.7368. **13.** 7.4747−10. **14.** 6.5999

189. Applications of logarithms. Since a logarithm is an exponent, its use furnishes a short cut to the solution of the following arithmetical operations:

I. Finding the product of several numbers, under the law of exponents expressed by the typeform $a^m \cdot a^n = a^{m+n}$.

II. Finding the quotient of one number divided by another under the typeform $a^m \div a^n = a^{m-n}$.

III. Finding a required power of a given number under the typeform $(a^m)^n = a^{mn}$.

IV. Finding a required root of a given number under the typeform $\sqrt[m]{a^n} = a^{\frac{n}{m}}$.

It is to be understood that the accompanying tables will give results approximately correct only to the fourth significant figure. If more than four places are required, more extensive tables may be used and the answer obtained correct to any reasonable number of places. The processes, however, are identical with those followed in using the accompanying four place tables.

Illustrative examples.

1. Multiply 277 by 383.

Add their logarithms. (277 times $383 = 10^{2.4425}$ times $10^{2.5832} = 10^{5.0257} = 106100$.)

log 277 = 2.4425
log 383 = 2.5832
antilog of 5.0257 = 106100. Ans.

2. Find 4.55·.00773·25.6.

Add their logarithms.

log 4.55 = 0.6580
log .00775 = 7.8893 − 10
log 25.6 = 1.4082
antilog of 9.9555 − 10 = .9026. Ans.

3. Divide 378 by 28.9.

Find the difference of the logarithms.

log 378 = 2.5775
log 28.9 = 1.4609
antilog of 1.1166 = 13.08. Ans.

10. Log $\sqrt[5]{\frac{a^3b^2}{\sqrt{c}}}=\frac{1}{5}\left[3\log a+2\log b-\frac{1}{2}\log c\right]$.

Complete the following logarithmic identities:

11. Log $\pi r^2=?$ 12. Log $\frac{4}{3}\pi r^3=?$ 13. Log $\sqrt{\frac{S}{4\pi}}=?$

14. Log $\sqrt{s(s-a)(s-b)(s-c)}=?$ 15. Log $\sqrt[3]{\frac{a^2-b^2}{c^2}}$.

Exercise 186. Problems

Solve by logarithms:

1. If a ray of light travels 186000 miles per second and it is 93,000,000 miles from the sun to the earth, how long does it take a ray of light to reach the earth from the sun?

2. The distance from the earth to the stars is measured by astronomers in light-years (the distance a ray of light travels in one year). How many miles is it to a star that is estimated as 8 light-years from the earth?

3. A certain bright star is estimated to be 300 light years from the earth. How many miles away is it?

4. What will \$5000 amount to if placed at 6% for 5 years, interest to be compounded annually?

Hint. One dollar at 6% for 5 years would amount to $(1.06)^5$.

5. What would be the amount, if the interest in the preceding problem were compounded semi-annually?

Hint. Use $(1.03)^{10}$.

6. What will \$50 amount to in 20 years at 7%, compounded annually?

7. In what time will \$1 double itself at 6%, compounded annually?

Hint. $(1.06)^x=2$, therefore $x(\log 1.06)=\log 2$ and $x=\frac{\log 2}{\log 1.06}$.

8. In what time will \$1 double itself at 6%, compounded semi-annually?

9. In what time will \$500 amount to \$750 at 6%, interest compounded annually?

10. What sum of money put at 6%, interest compounded semi-annually, will amount to \$10000 in 5 years?

11. What sum of money at 6%, interest compounded annually, will amount to \$10000 in 5 years?

12. The area of a circle is expressed by the formula $A = \pi r^2$. What is the area of a circle whose radius is 25 inches?

13. How many inches in the radius of a circle whose area is 1 square foot (144 sq. in.)?

14. The area of the equilateral triangle is expressed by the formula $A = \frac{s^2}{4}\sqrt{3}$. How many square inches in the area of an equilateral triangle whose side is one foot?

15. A regular hexagon is inscribed in a circle whose radius is 18 inches. What is the difference between their areas?

16. The apothem of a regular hexagon is $\frac{s}{2}\sqrt{3}$ where s is a side. Find the area of a regular hexagon whose apothem is 12 inches.

17. Find the radius of the circle circumscribed about a square whose side is 12 inches.

18. Find the radius of the circle circumscribed about a regular hexagon whose area is 144 square inches.

19. Find the apothem of the regular hexagon whose area is 144 square inches.

20. The area of the surface of a sphere is expressed by the formula $S = 4\pi r^2$. What is the area of the surface of a ball 10 inches in diameter?

21. How many square miles are on the surface of the earth assuming it to be a sphere whose radius is 3960 miles?

22. How many square miles are on the surface of the sun if its radius is approximately 433000 miles?

23. How many inches in the radius of the sphere whose area is 144 square inches?

24. The volume of a sphere is expressed by the formula $V=\frac{4}{3}\pi r^3$. What is the volume of a sphere whose radius is 12 inches?

25. What is the radius of the sphere whose volume is 1 cubic foot?

26. How many cubic miles are in the volume of the earth?

27. How many cubic miles are in the volume of the sun?

28. A cubic mile of water weighs how many tons, if one cubic foot of water weighs $62\frac{1}{2}$ pounds?

29. The volumes of two spheres have the ratio of the cubes of their radii. The volume of the sun is how many times the volume of the earth?

30. A cubic foot of lead will make how many spherical shot $\frac{1}{4}$ of an inch in diameter?

31. The area of a triangle whose sides are a, b, and c is expressed by the formula $A=\sqrt{s(s-a)(s-b)(s-c)}$ where $2s=a+b+c$. What is the area of the triangle whose sides are 17, 23, and 30 inches, respectively?

32. How many acres are in a triangular field whose sides are 755.5, 909.5, and 1325 feet, respectively?

33. What is the side of an equilateral triangle whose area is 1 square foot (144 square inches)?

34. Find the weight of a spherical ball of lead whose diameter is 8 in.

Note. Specific gravity is defined as the ratio of the weight of a body to the weight of an equal volume of water. For instance, the specific gravity of lead is 11.36 which means that a cubic foot of lead weighs 11.36 times 62.5 pounds, the weight of one cubic foot of water.

35. What would be the increase in the weight of the lead ball in No. 34 if 2 inches were added to its diameter?

36. Find the volume of a hemispherical cap of granite whose radius is 9 inches.

37. What is the weight of a cubic foot of gold, if its specific gravity is 19.27?

CHAPTER XVI

QUADRATIC EQUATIONS

191. Definitions. **A quadratic equation** is an equation of the second degree with respect to the unknown. If it contains no term of the first degree, it is called **a pure quadratic,** or an **incomplete quadratic.** If it contains terms of both the first and the second degree it is called an **affected quadratic** equation.

The equation $x^2 = 16$ is a pure quadratic. Likewise $x^2-5x=6$ and $x^2=3x$ are affected quadratic equations.

If an affected quadratic has one term that does not contain the unknown, it is called a **complete quadratic** equation. $x^2-5x=6$ is a complete quadratic as well as an affected quadratic.

Every quadratic equation in x can be reduced to the form $ax^2+bx+c=0$, where a may have any known value other than 0, and b and c may have any known values whatsoever.

As, for example, $(3-2x)(3+2x)=5-6x$ becomes $4x^2-6x-4=0$ when put in the form $ax^2+bx+c=0$.

192. Review. The student is already familiar with many of the processes of quadratic equations. However, a thorough review of the four types of problems in the following exercise will make the later work of the chapter easier to master.

Exercise 187

Solve the following equations by factoring:

1. $x^2-25=0$. **2.** $a^2-7a+12=0$.
3. $x^2-11x+24=0$. **4.** $4x^2-49=0$.
5. $3a^2+11a+8=0$. **6.** $5n^2-2n=7$.
7. $x^3-81x=0$. **8.** $13a=6a^2+6$.
9. $20=6n^2-7n$. **10.** $x^3-3x^2-54x=0$.
11. $3a^3=5a^2-2a$. **12.** $x^4-5x^2+4=0$.

Write the equations whose roots are the following:

13. 4, 5. **14.** -6, 3. **15.** -2, -7.
16. -7, 5. **17.** a, b. **18.** $-b$, c.
19. $2b$, $3c$. **20.** $-3a$, $4b$. **21.** $\frac{2}{3}$, $-\frac{1}{2}$.

Solution. $(x-\frac{2}{3})(x+\frac{1}{2})=0$ becomes $x^2-\frac{1}{6}x-\frac{1}{3}=0$ and, clearing of fractions, gives $6x^2-x-2=0$.

22. 2, $-\frac{1}{2}$. **23.** -3, $\frac{2}{3}$. **24.** $-\frac{2}{3}$, $-\frac{3}{2}$.
25. x, $-\frac{x}{3}$. **26.** $\frac{n}{3}$, $\frac{3n}{2}$. **27.** $2a$, $\frac{1}{2a}$.

Reduce the following pure quadratics to the form $x^2=a$ and then find the two roots by taking the square root of each member, as $x=\pm\sqrt{a}$:

28. $x^2=25$. **29.** $4x^2=9$. **30.** $x^2=7$.
31. $4x^2=15$. **32.** $9x^2=4a^2b^2$. **33.** $7x^2=8a$.

Supply the missing terms in the following trinomial squares:

34. $a^2+8a+?$ **35.** $n^2-10n+?$
36. $9a^2-6a+?$ **37.** $x^2+?+36a^2$.
38. $4n^2-?+9$. **39.** $x^2+x+?$
40. $4a^2-?+\frac{1}{4}b^2$. **41.** $9x^2-?+\frac{1}{4}$.
42. $9a^2-3a+?$ **43.** $\frac{4}{9}a^2-?+16y^2$.

193. Solution of a quadratic equation by completing the square.

First method. Complete the left member of the equation into a perfect trinomial square with the coefficient of x^2 positive 1.

Illustrative examples.

1. $x^2+8x=20$.

Solution.

$x^2+8x+16=36$. (Completing the square.)
$x+4=\pm 6$. (Extracting the square root of both members.)
$x=\pm 6-4$, $\therefore x=2$, and $x=-10$.
Both roots check.

2. $3x^2+7x=-3$.

Solution.

$x^2+\frac{7}{3}x=-1$. (Making the coefficient of x^2 positive 1.)

$x^2+\frac{7}{3}x+\frac{49}{36}=\frac{13}{36}$.

$x+\frac{7}{6}=\pm\frac{3.606}{6}$.

$x=\pm\frac{3.606}{6}-\frac{7}{6}$ $\therefore x=-.566$ and $x=-1.768$.

These roots are approximate. Why? Check for both.

Note. When the coefficient of x^2 is $+1$, the addition of the square of one-half the coefficient of x to both members transforms the left member into a perfect trinomial square.

Exercise 188

Solve the following quadratic equations by completing the square. If the roots are irrational simplify and reduce them to decimal form. Irrational roots may be checked in their radical form.

1. $n^2+6n=40$.	**2.** $a^2-12a=-20$.
3. $x^2-7x=18$.	**4.** $n^2+3n=40$.
5. $3x^2+8x+5=0$.	**6.** $5a^2+7a=6$.
7. $6a^2-11a-10=0$.	**8.** $x^2+6x=-7$.
9. $a^2-10a=-7$.	**10.** $3n^2+4n=9$.
11. $4=3a^2-2a$.	**12.** $5x^2+12x-17=0$.
13. $x^2+3x=5$.	**14.** $2x^2=5x+3$.
15. $n^2+2n=3$.	**16.** $8=3x^2-5x$.

Second method. (This method may be omitted at the discretion of the teacher and the problems of the exercise solved by the first method or by the formula.) Fractions may be avoided in completing the square by using some other number than 1 for the coefficient of x, as in the following:

Illustrative examples.

1. $3x^2+7x=-2$.

Solution.

$36x^2+84x=-24$. (Multiplying both members of the equation by 4 times the coefficient of x^2.)

$36x^2+84x+49=25$. (Adding to both members the square of the coefficient of x.)

$6x+7=\pm 5$. (Extracting the square root.)

$\therefore x=-\frac{1}{3}$ and $x=-2$.

2. $3x^2-4x=11$.

Solution.

$9x^2-12x=33$. (Multiplying both members by the coefficient of x^2.)

$9x^2-12x+4=37$. (Adding to both members the square of one-half the coefficient of x.)

$3x-2=\pm 6.083$. $\therefore x=2.694$ and $x=-1.361$.

Note. To avoid fractions in completing the square reduce the equation to the form $ax^2+bx=-c$ and multiply both members by $4a$ if b is an odd number, or by a if b is an even number. To complete the square, divide the coefficient of x by twice the square root of the coefficient of x^2 and add the square of the result to both members.

Exercise 189

Solve the following quadratic equations by the second method:

1. $2a^2+3a=9$.
2. $4n^2-3n=7$.
3. $2x^2-x=6$.
4. $6x^2-x-2=0$.
5. $3n^2-4n=7$.
6. $3n^2-4n=8$.
7. $a^2+3a=4$.
8. $5n^2+2n=3$.
9. $8=2x^2+3x$.
10. $(2-x)(2+x)=5x$.

194. Solution of quadratic equations by the formula. Since every quadratic equation can be reduced to the form $ax^2+bx+c=0$, where a, b, and c are the respective coefficients of x^2, x, and x^0 (or the term not containing x), it is evident

that if we solve this equation for x, we can use the resulting roots as a **formula** to obtain the roots of any quadratic equation.

Solution.

$$ax^2+bx+c=0.$$

$$ax^2+bx=-c.$$

$$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}.$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}. \qquad \therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a},$$

$$\text{or } x=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } x=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Illustrative examples.

1. $2x^2-x=6.$

Solution.

$2x^2-x-6=0.$ a is 2, b is -1, and c is -6.

Substituting in the formula. $x=\frac{1\pm\sqrt{1+48}}{4}=\frac{1\pm 7}{4}.$

$\therefore x=2$ and $x=-\frac{3}{2}.$ Both roots check.

2. $2x^2+3x+7=0.$

Solution. a is 2, b is 3, and c is 7.

$x=\frac{-3\pm\sqrt{9-56}}{4}.$ $\qquad x=\frac{-3+\sqrt{-47}}{4}$ and $x=\frac{-3-\sqrt{-47}}{4}.$

Check. (See § 182.)

Exercise 190

Solve the following quadratic equations by the formula:

1. $2x^2-3x+1=0.$

2. $3x^2-2x-1=0.$

3. $3x^2+x-2=0.$

4. $3x^2+8x+4=0.$

5. $7a^2-a-8=0.$

6. $2n^2-n-10=0.$

7. $2x^2+5x=3.$

8. $5a^2-a=6.$

9. $2a^2+3a=4.$

10. $3a^2+5a=7.$

11. $4a^2+5a+1=0.$

12. $2n^2+7n+6=0.$

13. $3a^2+6a+4=0.$ Ans. $a=\dfrac{-3\pm\sqrt{-3}}{3}.$

14. $x^2+3x+4=0.$ **15.** $2n^2-3n+2=0.$

16. $5a^2+5a+1=0.$ **17.** $2x^2+x+\frac{1}{2}=0.$

18. $x^3-1=0.$ Ans. $x=1, x=\dfrac{-1\pm\sqrt{-3}}{2}.$

Suggestion. By factoring $x-1=0$ and $x^2+x+1=0.$

19. $x^3+1=0.$ **20.** $x^3-8=0.$ **21.** $x^3+8=0.$

22. $x^4+x^2+1=0.$

Suggestion. $(x^2+x+1)(x^2-x+1)=0.$

23. $x^4-3x^2+9=0.$ **24.** $x^6-1=0.$

Solve the following equations for x:

25. $2n^2x^2+nx-3=0.$ **26.** $3x^2-5ax=2a^2.$

27. $2x^2-ax-10a^2=0.$ **28.** $ax^2+bx-a+b=0.$

29. $6abx^2-4a^2x-9b^2x+6ab=0.$

Suggestion. $6abx^2-(4a^2+9b^2)x+6ab=0.$

30. $acx^2-3bcx+2anx=6bn.$

31. $2x^2-\dfrac{11bx}{3a}-\dfrac{10b^2}{3a^2}=0.$ **32.** $\dfrac{ab-bx}{2x-a}=a-x.$

33. $\dfrac{2n+x}{2n-x}-\dfrac{2x-n}{2x+n}=\dfrac{8}{3}.$ **34.** $\dfrac{2x-a}{a+2x}+\dfrac{a+2x}{2x-a}=\dfrac{5}{2}.$

195. Character of the roots of a quadratic equation. The student will observe in the quadratic formula that—

(1) If $b^2-4ac=0$, both roots of the quadratic equation are $\dfrac{-b}{2a}$, and are therefore real and equal.

(2) If b^2-4ac is a perfect square, the roots are real, rational, and unequal.

(3) If b^2-4ac is a positive number but not a perfect square, the roots are real, irrational, and unequal.

(4) If b^2-4ac is a negative number, the roots are imaginary.

The expression b^2-4ac is called the **discriminant** of the quadratic equation.

An outline of numbers will be convenient in the work of the exercise that follows.

$$\text{Numbers}\begin{cases}\text{Real}\begin{cases}\text{Rational}\\ \text{Irrational}\end{cases}\\ \\ \text{Imaginary}\begin{cases}\text{Simple imaginary}\\ \text{Complex}\end{cases}\end{cases}$$

Exercise 191

In each of the following equations determine the value of b^2-4ac and, without solving, classify the roots:

1. $x^2-6x+8=0$. **2.** $4x^2-20x+25=0$.

3. $3x^2+5x-8=0$. **4.** $2x^2+3x-7=0$.

5. $x^2+x+1=0$. **6.** $2x^2+5x+4=0$.

7. In the equation $x^2-10x+k=0$, what must be the value of k in order that the roots may be equal?

8. Determine k in the equation $9x^2-kx+4=0$ so that the roots may be equal.

9. Find the greatest value m may have in $3x^2+2x+m=0$ so that the roots may be real.

10. Find the least value m may have in $2x^2+mx+2=0$ so that the roots may be real.

11. Find two factors of 1 whose sum is 3.

Suggestion. Let x and $\frac{1}{x}$ represent the factors.

12. Find two factors of 3 whose sum is 1.

13. Find the least value m may have in $3x^2+2x+m=0$ so that the roots may be complex.

14. One root of $x^3-2x^2-x-6=0$ is 3. Find the other roots.

Suggestion. One factor of x^3-2x^2-x-6 is $x-3$. Why?

15. One root of $x^3+3x^2+5x+6=0$ is -2. Find the other roots.

196. Relation of the roots and the coefficients of a quadratic equation. By dividing both members of a quadratic equation by the coefficient of x^2 and collecting terms in the left member, any quadratic equation may be put in the form $x^2+px+q=0$. This is called the p-form of the quadratic equation to distinguish it from $ax^2+bx+c=0$ which is called the a-form. It is evident that the a-form may be changed to the p-form and that $p=\frac{b}{a}$ and $q=\frac{c}{a}$.

Solving $x^2+px+q=0$ by the formula gives

$$x=\frac{-p+\sqrt{p^2-4q}}{2} \text{ and } x=\frac{-p-\sqrt{p^2-4q}}{2}.$$

The sum of these two roots is evidently $-p$ and the product is $\frac{(-p)^2-(\sqrt{p^2-4q})^2}{4}$ or q.

The two facts that the sum of the roots is $-p$ and the product q will be helpful in

(a) forming a quadratic equation when its roots are given,

(b) checking the roots resulting from the solution of a quadratic by determining if their sum is $-p$ and their product q.

Illustrative examples.

1. Write the equation whose roots are $5-\sqrt{7}$ and $5+\sqrt{7}$.

Solution.

Adding, $-p=10$ or $p=-10$.

Multiplying $q=18$.

Therefore $x^2-10x+18=0$ is the equation.

2. Does $x=\frac{4\pm\sqrt{-5}}{3}$ satisfy $3x^2-8x+7=0$?

Solution. The equation in the p-form is $x^2-\frac{8}{3}x+\frac{7}{3}=0$. Whence $p=-\frac{8}{3}$ and $q=\frac{7}{3}$. $\frac{4+\sqrt{-5}}{3}+\frac{4-\sqrt{-5}}{3}=\frac{8}{3}=-p$ and $\frac{4+\sqrt{-5}}{3}\cdot\frac{4-\sqrt{-5}}{3}=\frac{16-(-5)}{9}=\frac{21}{9}=\frac{7}{3}=q.$ Ans. Yes.

Exercise 192

Solve each of 1–10 *in two ways. Write the equations whose roots are:*

1. $5, -3$. **2.** $-7, -4$. **3.** $3, -\frac{2}{3}$.

4. $\frac{8}{3}, \frac{3}{2}$. **5.** $3-\sqrt{7}, 3+\sqrt{7}$.

6. $\frac{7 \pm \sqrt{5}}{2}$. **7.** $\frac{5 \pm \sqrt{3}}{3}$. **8.** $\frac{4 \pm \sqrt{-3}}{2}$.

9. $\frac{3 \pm \sqrt{-2}}{2}$. **10.** $\frac{-5 \pm \sqrt{-3}}{2}$.

11. Does $x=\frac{3 \pm \sqrt{-7}}{4}$ satisfy $2x^2-3x+2=0$?

12. Does $x=\frac{1+3\sqrt{-1}}{2}$ satisfy $2x^2-2x+5=0$?

13. One root of $x^2+8x+k=0$ is 2. Find the other root and the value of k.

Note. No. 13 can be done in two ways: $x=2$ must satisfy the equation, and the sum of the roots is -8. Why?

14. One root of $3x^2-5x+n=0$ is $\frac{2}{3}$. Find the other root and n.

15. One root of $x^2-kx+20=0$ is 4. Find the other root and k.

16. Find m and n in the equation $x^2+mx+n=0$ so that the roots may be -5 and -8.

17. One root of $x^2-11x+k=0$ is 3 more than the other. Find the roots and k.

Suggestion. What is the sum of the roots?
How can you find two numbers whose sum is 11 when one is 3 more than the other?

18. One of the roots of $x^2-9x+k=0$ is twice the other. Find the roots and k.

197. Equivalent equations. Equations that involve the same unknown number and have the same roots are called **equivalent equations.** As, for example, $\frac{x+4}{x}=\frac{x-2}{x-3}$ and $2x=8$ are equivalent equations, for $x=4$, satisfies both and no other value of x satisfies either. In the solution of a quadratic equation (or of any other type) it is necessary that the equations obtained in the successive steps of the solution be equivalent, each to the original and therefore to one another, otherwise roots may be introduced or roots may be lost.

198. Extraneous roots and lost roots. When a root is obtained in the solution of an equation that does not check in the original equation, it is called an **extraneous root.**

A study of the following **illustrative examples** will serve to show how roots may be introduced or lost in the solution of equations.

1. The equation $2x-3=x+2$ is satisfied by $x=5$.

If each member of this equation is multiplied by $x-3$, we get $2x^2-9x+9=x^2-x-6$, which becomes $x^2-8x+15=0$, or $(x-5)(x-3)=0$.

Therefore $x=3$ and $x=5$. But $x=3$ does not check in the original equation and is an extraneous root.

2. The equation $2x^2-10x+12=x^2-5x+6$ is satisfied by $x=2$ and $x=3$.

If each member is divided by $x-3$, we get $2x-4=x-2$, which becomes $x-2=0$. Therefore $x=2$. The root $x=3$ has been lost.

3. Solve the equation $\frac{3x}{x-3}+x+1=\frac{9}{x-3}$.

Clearing of fractions gives $3x+x^2-2x-3=9$, or $x^2+x-12=0$ which is satisfied by $x=3$ and $x=-4$.

But $x=3$ does not check in the original equation and is extraneous. The root $x=3$ was introduced when the original equation was cleared of fractions by multiplying each member by $x-3$. If the solution had been as follows no extraneous root would have entered:

$$\frac{3x}{x-3}+x+1=\frac{9}{x-3}. \qquad \frac{3x}{x-3}-\frac{9}{x-3}+x+1=0.$$

$$\frac{3x-9}{x-3}+x+1=0. \qquad 3+x+1=0 \text{ and } x=-4.$$

In the solution of any type of equations it is best to keep in mind (1) that checking will detect an extraneous root, (2) that no factor containing the unknown can be removed without removing a root.

199. Irrational equations resulting in quadratics. Many irrational equations reduce to the quadratic form in the process of solution. Often one of the resulting roots is extraneous and care must be used in checking both.

Illustrative examples.

1. $\sqrt{3x+1}=x-1$.

Solution. $3x+1=x^2-2x+1$. (Squaring both members.)

Whence $x^2-5x=0$ and $x(x-5)=0$, or $x=5$ and $x=0$.

Checking. If $x=5$, $\sqrt{15+1}=5-1$ and $4=4$, which checks.

If $x=0$, $\sqrt{0+1}=0-1$ and $\sqrt{1}=-1$ which does not check.

2. $\sqrt{x+5}+\sqrt{x}=\sqrt{6x+1}$.

Squaring both members, $x+5+2\sqrt{x^2+5x}+x=6x+1$.

Combining, $2\sqrt{x^2+5x}=4x-4$.

Dividing by 2, $\sqrt{x^2+5x}=2x-2$.

Squaring, $x^2+5x=4x^2-8x+4$. Whence $3x^2-13x+4=0$.

Therefore $(3x-1)(x-4)=0$ and $x=4$, $x=\frac{1}{3}$.

Checking. If $x=4$, $\sqrt{9}+\sqrt{4}=\sqrt{25}$, and $3+2=5$. Checks.

If $x=\frac{1}{3}$, $\sqrt{\frac{16}{3}}+\sqrt{\frac{1}{3}}=\sqrt{3}$, and $\frac{4}{3}\sqrt{3}+\frac{1}{3}\sqrt{3}=\sqrt{3}$. Does not check.

Exercise 193

Solve and check the following irrational equations:

1. $\sqrt{6x-5}=2x-5$.
2. $\sqrt{2a^2+2a+1}=a+2$.
3. $\sqrt{3n+3}=4-\sqrt{n-1}$.
4. $\sqrt{2x-5}-\sqrt{x-6}=2$.
5. $\sqrt{n+4}+\sqrt{2n-1}=6$.
6. $\sqrt{m-4}=\sqrt{4m+5}-\sqrt{3m+1}$.
7. $\sqrt{2a-3}-\sqrt{3a-4}=\sqrt{a-1}$.
8. $\sqrt{x^2+4\sqrt{7x+15}}=x+2$.
9. $\sqrt[3]{x^3-5x^2-2x+2}=x-1$.
10. $\sqrt{25+4x}-\sqrt{8+x}-\sqrt{2x+9}=0$.
11. $\sqrt{25+4x}+\sqrt{8+x}-\sqrt{2x+9}=0$.
12. $\sqrt{4x-6n}=\sqrt{3x+2n}-\sqrt{2n}$. (Solve for x.)

200. Solution of equations of higher degree. The complete solution of all types of equations of higher degree than the second is beyond the realm of elementary algebra, and even the processes of advanced mathematics fail in many cases. However, there are some types that can be solved by applying the principles already learned.

Illustrative examples.

Case I. Equations that may be reduced by factoring.

1. $x^3-64=0$.

Factoring $(x-4)(x^2+4x+16)=0$.
Solving $x-4=0$ and $x^2+4x+16=0$,
$\therefore x=4$ and $x=-2\pm2\sqrt{-3}$.

Note. All roots should be checked by substituting in the original equation. It will be observed that any equation of third or higher degree can be solved if it can be reduced by factoring to two or more equations that are linear or quadratic.

Case II. Equations that are quadratic in form with respect to some expression containing the unknown.

2. $x^4+3x^2+1=0$.

This is quadratic with respect to x^2.

Then $x^2=\dfrac{-3\pm\sqrt{5}}{2}$.

Whence $x=\pm\sqrt{\dfrac{-3\pm\sqrt{5}}{2}}=\pm\frac{1}{2}\sqrt{-6\pm2\sqrt{5}}$.

3. $(x^2+5x)^2-5(x^2+5x)=6$.

Solution.

Put $x^2+5x=n$. Then $n^2-5n=6$.
Whence $n=6$ and $n=-1$.
Then $x^2+5x=6$. $\therefore x=-6$, and $x=1$.

Also $x^2+5x=-1$. $\therefore x=\dfrac{-5\pm\sqrt{21}}{2}$.

4. $x^2+2x-6\sqrt{x^2+2x+10}=-15$.

Solution.

$x^2+2x+10-6\sqrt{x^2+2x+10}=-5$. (Adding 10 to each number.)
Put $\sqrt{x^2+2x+10}=n$. Then $n^2-6n=-5$.
Whence $n=1$ or $n=5$.
Then $\sqrt{x^2+2x+10}=1$. $\therefore x^2+2x+10=1$.
Whence $x=-1\pm2\sqrt{-2}$.
$\sqrt{x^2+2x+10}=5$. $\therefore x^2+2x+10=25$.
Whence $x=3$ and $x=-5$. Check.

Exercise 194

Find all the roots of the following equations and check:

1. $x^3-27=0$.
2. $(x-3)\ (2x^2+7x+2)=0$.
3. $(n^2-n+1)\ (n^2+n+1)=0$.
4. $x^4-9x^2+20=0$.
5. $a^6-64=0$.
6. $x-x^{\frac{1}{2}}=20$.
7. $3a^{\frac{2}{3}}+4a^{\frac{1}{3}}=4$.
8. $\sqrt{x}-3\sqrt[4]{x}=40$.
9. $2x+3-5\sqrt{2x+3}=-6$.
10. $x^3-4x^{\frac{3}{2}}+3=0$.
11. $(a-2)^3-3(a-2)^{\frac{3}{2}}=40$.
12. $a^2+3a+3\sqrt{a^2+3a}=10$.
13. $x^4-7x^2+1=0$.

Note. No. 13 can be solved in two ways. First by substituting n for x^2 in the equation, second, by factoring. Solve by both methods and show that the roots obtained by one method agree with those obtained by the other method.

14. $x^4+3x^2+4=0$.
15. $4x^4-x^2+4=0$.
16. $x^{-2}-5x^{-1}+6=0$.
17. $6x^{-2}+6=13x^{-1}$.
18. $2\left(x+\dfrac{1}{x}\right)^2-9\left(x+\dfrac{1}{x}\right)+10=0$.
19. $\sqrt{\dfrac{3-5x}{3-x}}+\sqrt{\dfrac{3-x}{3-5x}}=\dfrac{5}{2}$.
20. $\dfrac{17}{4}-\dfrac{x+3}{3x-7}=\dfrac{3x-7}{x+3}$.

201. Use of the quadratic equation. When a quadratic equation is used in the solution of a problem, care is necessary not only in checking the roots of the equation but also in determining whether both of these roots satisfy the conditions imposed by the problem. As, for instance, the length of a rectangle cannot be a negative number.

Exercise 195. Problems

1. Find two consecutive numbers whose product is 90.
2. Find two consecutive odd numbers the sum of whose squares is 130.
3. Find three consecutive numbers the sum of whose squares is 110.
4. Separate the number 20 into two parts, the sum of whose squares shall be 202.

25. A rectangular piece of tin, 6 inches longer than it is wide, is made into an open box by cutting a 3 inch square from each corner and turning up the sides. If the volume of the box is 216 cubic inches, find the dimensions of the original piece of tin.

26. A certain number exceeds 4 times its square root by 12. Find the number.

27. A man bought a number of sheep for $250. Four of the sheep died and he sold the others for $2.50 per head more than he paid and received $240. How many sheep did he buy?

28. A landscape gardener has a flower bed that is 12 yards wide and 18 yards long. He wishes to double the area of the flower bed by increasing the width and length the same amount. Find the number of yards the dimensions must be increased.

29. A farmer starts to cut a field of grain, 50 rods by 60 rods, by driving his reaper round and round the field. Find the width of the strip that is cut when two-thirds of the field remains uncut.

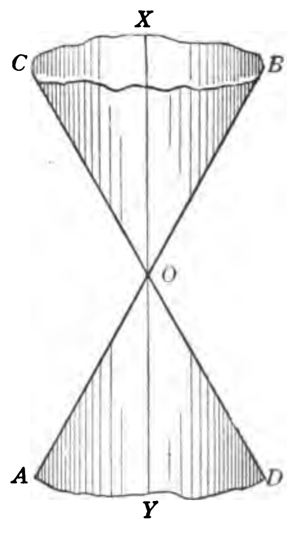
CHAPTER XVII

QUADRATIC EQUATIONS THAT INVOLVE TWO VARIABLES

202. Quadratic equations as conic sections. If the graph of any quadratic equation in two variables be drawn, we have a curved line belonging to one of the four groups of curves known as **conic sections.** These curves are the **circle,** the **parabola,** the **hyperbola,** and the **ellipse.** The definitions of these are as follows:

A **circle** is the **locus** (or path) of a point that is always at a given distance from a fixed point.

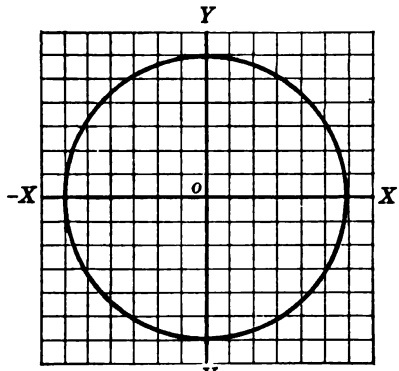
A **parabola** is the locus of a point whose distance from a fixed point, called the **focus,** is always equal to its distance from a fixed line, called the **directrix.**



A **hyperbola** is the locus of a point the difference of whose distances from two fixed points, called the **foci,** is always equal to a constant distance.

An **ellipse** is the locus of a point, the sum of whose distances from two fixed points, called the **foci,** is always equal to a constant distance.

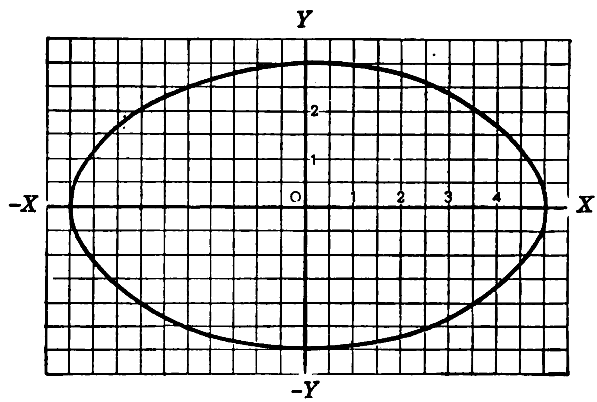
203. Conic sections. Suppose the straight lines AB and CD intersect at O and that XY is the



Circle $x^2 + y^2 = 36$

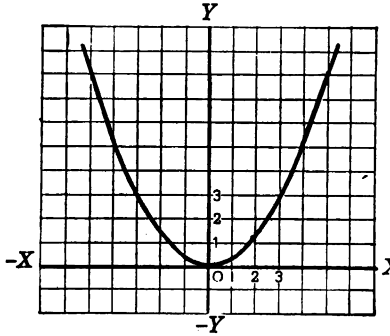
bisector of two opposite angles. Now if the whole figure be revolved about XY as an axis, the lines AB and CD will generate (or form) a **conical surface.** This surface will have two parts, or **nappes,** meeting point to point. The conical surface will be indefinite, or unlimited, in extent, since AB and CD are lines of unlimited length. Next, suppose the conical surface to be intersected by a plane surface. If the plane is perpendicular to XY, the intersection will be a circle.

If the plane is oblique to XY and cuts completely through one nappe, the intersection will be an ellipse.



Ellipse $9x^2 + 25y^2 = 225$

If the plane is parallel to one of the lines AB or CD in any of its positions, it will cut one nappe only and the

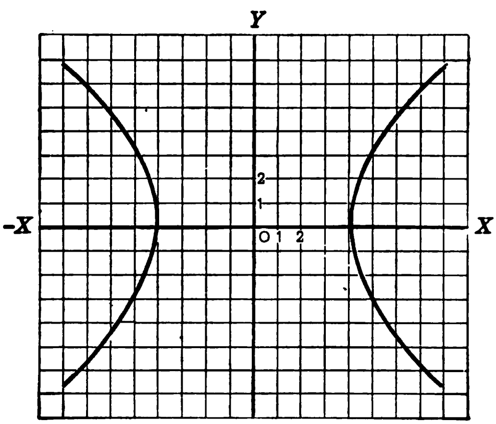


Parabola $x^2 = 3y$

intersection will be a parabola. If the plane cuts both of the nappes, the intersection will be the double open curve, or hyperbola.

Note. If a quadratic equation of two unknowns can be factored into two linear equations, the graph in this case will be a pair of straight lines, or the respective graphs of the two linear equations.

204. Graphing a quadratic equation in two variables. In graphing a quadratic equation in two variables, it is best to solve the equation for one variable in terms of the other and substitute numbers, both positive and negative, for the second variable until a sufficient number of sets of values has been found to locate the curve.



Hyperbola $x^2 - y^2 = 16$

Illustrative example.
Graph $x^2 - y^2 = 16$.

Solution.
Solving for x, gives $x = \pm\sqrt{16+y^2}$.

Substituting values for y, gives the following sets:

When $y =$	0,	2,	-2,	3,	-3,	4,	-4,
then $x =$	± 4,	± 4.5,	± 4.5,	± 5,	± 5,	± 5.6,	± 5.6.

Notice that there are two sets of values for each value of y, for when $y = 2$, then $x = +4.5$ and $x = -4.5$.

See the table of square roots following Chapter XXII.

Exercise 196

Graph the following equations:

1. $y=\frac{12}{x}$.
2. $y=\frac{12}{x}-3$.
3. $x^2-y^2=25$.
4. $2x^2-5y^2=50$.
5. $x^2+y^2=20$.
6. $2x^2+3y^2=30$.
7. $(x+3)^2+y^2=36$.
8. $(x+2)^2+(y-3)^2=25$.
9. $y=x^2$.
10. $y=x^2+3$.
11. $y=x^2-6x+9$.
12. $y=x^2-6x-3$.
13. $x=y^2$.
14. $x=y^2+7$.
15. $y=x^2+2x$.
16. $y=2x^2+4x$.

Graph each of the following pairs of equations and write the values of the points of intersection:

17. $x^2+y^2=29$
    $xy=10$.
18. $3x^2+2y^2=35$
    $x^2-2y^2=1$.
19. $x+xy=35$
    $y+xy=32$.
20. $x^2+y\ =7$
    $x\ +y^2=11$.

Note. The student will observe at this point that he can recognize certain types of curves from the appearance of the equations, for example,

(1) The curve $x^2+y^2=r^2$ is a circle whose center is at the origin and whose radius is r.

(2) The curve $(x-a)^2+(y-b)^2=r^2$ is a circle whose center is at the point $(x=a,\ y=b)$ and whose radius is r.

(3) The curve $x^2-y^2=a$ is a hyperbola which cuts the X axis but not the Y axis; likewise the curve $y^2-x^2=a$ cuts the Y axis but not the X axis.

(4) The curve $xy=\pm a$ is a hyperbola which does not touch either axis.

(5) The curve $ax^2+by^2=c$ is an ellipse.

(6) The curve $y=ax^2+bx+c$ is a parabola.

205. Solution of quadratic equations of one unknown by graphing. We cannot graph equations of one unknown but we can solve equations of the type $ax^2+bx+c=0$ by graphing the equation $y=ax^2+bx+c$. For example, to solve $x^2-2x-15=0$, we graph $y=x^2-2x-15$. Now, we notice that when

$y=0$, the values of x will be the roots of $x^2-2x-15=0$. This method is of use only when the roots are real. If we attempt to solve $x^2+3x+7=0$ by graphing $y=x^2+3x+7$, we will get a parabola that does not cut the X axis, which means, of course, that when $y=0$, there are no real values of x.

Exercise 197

Solve the following quadratic equations by graphing:

1. $x^2-x-12=0$.
2. $x^2-3x-40=0$.
3. $2x^2-7x+3=0$.
4. $2x^2-7x-15=0$.
5. $x^2-9=0$.
6. $x^2=16$.
7. $x^2=20$.
8. $3x^2-7x=6$.
9. $2x^2+3x=20$.
10. $6x^2+7x=5$.

206. Solution of systems of simultaneous quadratic equations.

Not all sets of simultaneous quadratic equations that involve the same unknowns can be solved by the ordinary processes of algebra. However, there are a number of methods, each of which will solve exercises of a particular type. Only the more important types are presented in this chapter.

Case I. A pair of equations, one linear and one quadratic. All of this type can be solved by the method of substitution.

Illustrative example.

(1) $3x+2y=7$.
(2) $2x^2+xy=4$.

From (1) $x=\dfrac{7-2y}{3}$.

Substituting in (2), $\dfrac{2(49-28y+4y^2)}{9}+\dfrac{y(7-2y)}{3}=4$.

Simplifying, $2y^2-35y+62=0$.
Whence $y=2$ or $y=15\frac{1}{2}$.
Substituting in (1) when $y=2$, $x=1$; when $y=15\frac{1}{2}$, $x=-8$.
Check both sets in each equation.

Case II. When all terms of each equation that contain the unknowns are of the second degree. There are several methods that are satisfactory for the solution of exercises of this type. The method of eliminating the constant terms is the only one that will be introduced into the work of this chapter.

Illustrative example.

(1) $2a^2-ab=12$.

(2) $a^2+ab-3b^2=3$.

$$\begin{array}{ll} \text{Multiply (2) by 4,} & 4a^2+4ab-12b^2=12. \\ \text{Subtract (1)} & \underline{2a^2-\ \ ab \qquad\ =12} \\ & 2a^2+5ab-12b^2=\ 0. \end{array}$$

Factoring $(2a-3b)(a+4b)=0$.

Note. Show that we now have two examples under Case I.

Whence $a=\frac{3b}{2}$, $a=-4b$.

Substituting $a=\frac{3b}{2}$ in (1) $\frac{9b^2}{2}-\frac{3b^2}{2}=12$.

Whence $b=\pm2$. Therefore $a=\frac{3b}{2}=\pm3$.

Substituting $a=-4b$ in (1) $32b^2+4b^2=12$.

Whence $b=\pm\frac{1}{3}\sqrt{3}$. Therefore $a=-4b=\pm\frac{4}{3}\sqrt{3}$.

Therefore when $a=\pm3$, $b=\pm2$; when $a=\pm\frac{4}{3}\sqrt{3}$, $b=\pm\frac{1}{3}\sqrt{3}$.

All four sets of answers should be checked in each equation.

Note. When double answers are paired as above, it is understood that when $a=+3$, $b=+2$, and when $a=-3$, $b=-2$.

Case III. When some combination of the two given equations is possible that will make a new equation with the left member a perfect square.

Illustrative example.

(1) $x^2+y^2=17$

(2) $\quad xy=4$

Adding twice (2) to (1), $x^2+2xy+y^2=25$.

Extracting sq. rt., $x+y=\pm5$.

Subtracting twice (2) from (1), $x^2-2xy+y^2=9$.

Extracting sq. rt., $x-y=\pm3$.

Combine $\quad \underline{x+y=\pm5.}$

Note. There are four different ways of combining the two equations, giving four sets of roots. When $x=\pm4$, $y=\pm1$; when $x=\pm1$, $y=\pm4$. Check completely.

Case IV. When the members of one equation are exactly divisible by the corresponding members of the other equation.

Illustrative example.

(1) $x^3+y^3=35$.

(2) $x^2-xy+y^2=7$.

Dividing (1) by (2), $x+y=5$. (3)

Solving (2) and (3) by the method of Case I, when $x=3$, $y=2$; when $x=2$, $y=3$. Check.

Case V. When one of the equations can be solved for some expression containing the unknowns.

Illustrative example.

(1) $x^2+y^2=25$.

(2) $(x+y)^2-12(x+y)+35=0$.

Solving (2) for $(x+y)$, $x+y=7$, or $x+y=5$.

Solving $x+y=7$ with $x^2+y^2=25$; when $x=4$, $y=3$; when $x=3$ $y=4$.

Solving $x+y=5$ with $x^2+y^2=25$; when $x=5$, $y=0$; when $x=0$, $y=5$. Check all sets.

Case VI. Combining the processes of the preceding cases. Many sets of simultaneous equations can be solved by using two or more of the preceding methods, or by applying some simple process of algebra already learned.

Illustrative examples.

I. (1) $x^2+xy+x=18$.

(2) $y^2+xy+y=12$.

Suggestion. Add (1) and (2) and solve for $(x+y)$.

Ans. When $x=3$, $y=2$; when $x=-\frac{18}{5}$, $y=-\frac{12}{5}$.

II. (1) $xy+x-y=7$.

(2) $xy(x-y)=12$.

Suggestion. Put $xy=a$, $x-y=b$.

Then, $a+b=7$ and $ab=12$. Solve for a and b and equate their values with xy and $x-y$. Ans. When $x=4$, $y=1$: when $x=-1$, $y=-4$.

When $x=\pm\sqrt{7}+2$, $y=\pm\sqrt{7}-2$.

III. (1) $5x^2-3xy=-3$.

(2) $2x^2+5xy=36$.

Suggestion. Multiply (1) by 5, (2) by 3, and eliminate xy by addition. Ans. When $x=\pm\sqrt{3}$, $y=\pm 2\sqrt{3}$. (Two sets.) Check.

Exercise 198

Solve and check the following:

1. $x+y=9$
$x^2+y^2=41.$

2. $a+b=7$
$ab=12.$

3. $m^2-n^2=45$
$m-n=5.$

4. $x+y=3$
$x^2+y^2=45.$

5. $2x+2y=5$
$xy=1.$

6. $3x+2y=8$
$3x^2+2y^2=14.$

7. $(x-3)\ (y+2)=0$
$x+2y=1.$

8. $xy+4x-2y=8$
$x+y=5.$

9. $2x-3y=9$
$xy=-3.$

10. $2x-3y=6$
$x^2-3xy=0.$

11. $2x+2y=3$
$2x^2-5xy+2y^2=0.$

12. $x^2-xy=54$
$xy-y^2=18.$

13. $x^2+y^2=13$
$xy=6.$

14. $y^2+15=2xy$
$x^2+y^2-21=xy.$

15. $5x^2-y^2=1$
$3y^2=xy+10.$

16. $x^2+xy+2y^2=74$
$2x^2+2xy+y^2=73.$

17. $(x-y)\ (x+y)=40$
$(3y+x)\ (3x+y)=384.$

18. $m^2+n^2+m+n=18$
$mn=6.$

19. $a^2b^2+ab=6$
$a^2+b^2=5.$

20. $(x+y)^2+(x+y)=30$
$xy=6.$

21. $x^2+y^2=25$
$xy+x+y=19.$

22. $x^2+4y^2=13$
$xy+x+2y=-2.$

23. $a^3-b^3=26$
$a-b=2.$

24. $a^3+8b^3=16$
$a+2b=4.$

25. $x^3+y^3=91$
$x+y=7.$

26. $x^4+x^2y^2+y^4=21$
$x^2+xy+y^2=7.$

27. $x^4+x^2y^2+y^4=481$
$x^2+xy+y^2=37.$

28. $m^3+n^3=2a^3+6a$
$m^2-mn+n^2=a^2+3.$

29. $r^3+s^3=35$
$r^2-rs+s^2=7.$

30. $\dfrac{1}{x^2}-\dfrac{1}{y^2}=\dfrac{5}{36}$
$\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{6}.$

31. $m^4+m^2n^2+n^4=133$
$m^2+mn+n^2=19.$

32. $a^2+b^2=3+ab$
$a^4+b^4=21-a^2b^2.$

33. $a^2-ab+2b^2=16$
$2a^2+3ab-2b^2=8.$

34. $3x^2+y^2=37$
$y^2=29+2x-2x^2.$

35. $a^2+b^2=3ab-1$
$a-b=ab-7.$

36. $3x^2+2y^2=30$
$5x^2-3y^2=-7.$

37. $2x^2+3xy=36$
$3x^2-2xy=15.$

38. $x^2+2y^2+x+2y=10$
$x^2-y^2+x-y=4.$

39. $x+2xy=-6$
$3y-2xy=2.$

40. $4R^2+r^2+4R+2r=6$
$2Rr=1.$

41. $x^2+xy+y^2-63=0$
$x-y+3=0.$

42. $\frac{6}{x}=\frac{y}{10}$
$x-y=11.$

43. $\frac{x}{y}=2$
$xy=8.$

44. $x^2+y^2=500$
$\frac{x+y}{x-y}=3.$

45. $xy+y^2=4$
$2x^2+3xy=27.$

46. $a^4+a^2b^2+b^4=21$
$a^2+ab+b^2=7.$

47. $\frac{y}{x}+\frac{x}{y}=\frac{5}{2}$
$x^2+y^2=5.$

48. $2a^2-3ab+2b^2=43$
$a^2-ab+b^2=39.$

49. $\frac{1}{a}+\frac{1}{b}=\frac{3}{2}$
$\frac{1}{a^2}+\frac{1}{b^2}=\frac{5}{4}$

50. *Solve for x and y:*
$x^2+xy=ab$
$xy+y^2=a^2-ab.$

Exercise 199. Problems

1. Find two numbers whose sum is 2 and the sum of whose squares is 34.

2. The sum of two numbers multiplied by the greater is 28 and the difference of the two numbers is 1. Find the numbers.

3. The difference of two numbers is 2 and the difference of their cubes is 98. Find the numbers.

4. The area of a certain rectangle is 80 square inches. If the width is increased 3 inches and the length is decreased 3 inches, the area will be increased 24 square inches. Find the dimensions.

5. The sum of two numbers is a and the sum of their squares is b. Find the numbers and check the results.

6. The sum of the squares of two numbers exceeds their product by 39, and the difference of the numbers is 33 less than their product. Find the numbers.

7. The sum of two numbers equals the difference of their squares, and 3 times their product equals 4 times the square of the smaller. Find the numbers.

8. The sum of the squares of two numbers added to their sum is 14, and the difference of the squares of the two numbers added to their difference is 10. Find the numbers.

9. The diagonal of a certain rectangle is 13 inches and the area is 60 square inches. Find the dimensions of the rectangle.

10. If a certain number of two digits is divided by the product of the digits, the quotient is 2. If the number is divided by the sum of the digits, the quotient is 4. Find the number.

11. Separate 8 into two parts such that the sum of the cubes of the parts shall be 152.

12. The sum of the squares of two numbers is 170. If the smaller number was 3 greater and the larger number 1 less, the sum of their squares would be 200. Find the numbers.

13. The hypotenuse of a right triangle is 26 inches and the area is 120 square inches. Find the two sides.

14. Some boys hired a motor boat for a trip for $36. Three of the boys were not able to go on which account it cost each of the others $1 more. Find the number of boys that went on the trip and what each one paid.

15. If the average speed of a railway train was increased 10 miles per hour, it would require 2 hours less time to go 400 miles. Find the rate of the train and the time required to make the trip.

16. The three sides of a triangle are 10, 12, and 14 inches. Find the projections upon the longest side of the other two sides.

Suggestion. Draw the altitude to the longest side and study the two right triangles. Let x represent the numerical length of the altitude and y and $14-y$, the segments of the base.

17. Find the area of the triangle of No. 16.

18. Find the area of a triangle whose sides are 8, 9, and 10 inches.

19. A boy lodges his kite in the top of a tree. He stretches the string to a point 14 feet from the foot of the tree and then to a point 64 feet from the foot of the tree. He finds that it requires 30 feet more string to reach the second point. Find the height of the tree.

20. The difference of the squares of two numbers is a, and the quotient obtained by dividing the sum of the numbers by their difference is b. Find the numbers.

21. A guy wire is attached to the top of a flagpole and to a stake in the ground 14 feet from the base of the pole. It is observed that the wire must be lengthened 10 feet to reach a stake 36 feet from the base of the pole. Find the length of the wire and the height of the pole.

CHAPTER XVIII

RATIO, PROPORTION, AND VARIATION

207. Definitions. Define the following: ratio, antecedent, consequent, proportion, terms of a. proportion, means, extremes, mean proportional, third proportional, and fourth proportional.

208. Fundamental laws of proportion.

I. In any proportion the product of the means equals the product of the extremes. That is, if $\frac{a}{b}=\frac{c}{d}$, then $ad=bc$.

II. If the product of two factors equals the product of two others, either two may be made the means and the other two the extremes of a proportion. That is, if $ad=bc$, then $\frac{a}{b}=\frac{c}{d}$.

209. Important principles of proportion.

I. If four quantities are in proportion, they are in proportion by alternation. That is, if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{c}=\frac{b}{d}$.

Explain by applying the two fundamental laws.

II. If four quantities are in proportion, they are in proportion by inversion. That is, if $\frac{a}{b}=\frac{c}{d}$, then $\frac{b}{a}=\frac{d}{c}$. Explain.

III. If four quantities are in proportion, they are in proportion by composition. That is, if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$.

IV. If four quantities are in proportion, they are in proportion by division. That is, if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a-b}{b}=\frac{c-d}{d}$.

210. Geometrical principles.

I. Corresponding sides or altitudes of similar triangles are in proportion.

II. Corresponding sides or diagonals of similar polygons are in proportion.

III. If a line is drawn through two sides of a triangle parallel to the third side, it divides the two sides proportionally.

IV. If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse, the perpendicular is a mean proportional between the segments of the hypotenuse.

V. The bisector of an angle of a triangle divides the opposite side into segments that are proportional to the adjacent sides of the angle.

VI. If, from a point without a circle, a tangent and a secant are drawn, the tangent is a mean proportional between the whole secant and its external segment.

VII. The areas of two similar polygons have the same ratio as the squares of any two corresponding sides.

VIII. A line is said to be divided in extreme and mean ratio when the longer part is a mean proportional between the whole line and the shorter part.

Exercise 200

Write the following ratios as fractions and simplify:

1. $32 : 48$. **2.** $57 : 95$. **3.** $3\frac{5}{8} : 4\frac{1}{7}$.

4. $9a^2b^3 : 21a^5b^2$. **5.** $x^2-y^2 : (x-y)^2$.

6. $a^3+b^3 : a^4+a^2b^2+b^4$. **7.** $\dfrac{x^4-y^4}{x^2-xy+y^2} : \dfrac{x^2+y^2}{x^3+y^3}$.

Arrange the following proportions by alternation:

8. $\dfrac{a}{b}=\dfrac{c}{d}$. **9.** $\dfrac{x-y}{2x}=\dfrac{3y}{x+y}$. **10.** $\dfrac{a^2}{c^2}=\dfrac{b^2}{d^2}$.

Arrange the following proportions by composition and simplify:

11. $\frac{x-y}{y}=\frac{a-b}{b}$. 12. $\frac{2x-3}{8}=\frac{3x-5}{7-3x}$. 13. $\frac{a-b}{a+b}=\frac{c-d}{c+d}$.

Find the value of x in each of the following proportions:

14. $\frac{x}{3}=\frac{x+8}{9}$. 15. $\frac{8}{x}=\frac{x}{32}$. 16. $\frac{7}{x}=\frac{x}{14}$. 17. $\frac{x+3}{x-5}=\frac{x+7}{x-11}$.

18. $\frac{2x+3}{3x-2}=\frac{x+6}{2x+1}$. 19. $\frac{3x+5}{5x-2}=\frac{2x+3}{3x-1}$. 20. $\frac{3-x}{6+x}=\frac{4-x}{8+x}$.

Find the mean proportional between:

21. 7 and 28. 22. 9 and 3. 23. $\frac{1}{2}$ and $\frac{1}{4}$.

24. a^3b and ab^5. 25. a^2-b^2 and $a+b$.

26. m^3-n^3 and m^4-n^4.

27. Given $xy=mn$, form a proportion from x, y, m, and n.

28. Given $a^2-b^2=x^4+x^2+1$, form a proportion.

29. The ratio of 625 to x^3 is 5. Find x.

30. The mean proportional between 5 and 9.6 is x. Find x.

Exercise 201. Problems

1. What number must be subtracted from 50, 45, 75, and 65 so that the remainders shall form a proportion?

2. The ratio of the rate of a local train to that of an express train is 2 : 3. If the local train runs 30 miles per hour, find the rate of the express train.

3. The ratio of the length of Lake Erie to the length of Lake Michigan is 3 : 4. What is the length of each if Lake Michigan is 90 miles longer than Lake Erie?

4. A base ball player threw a ball 210 feet. The ratio of this distance to the distance usually considered the limit for players is 5 : 7. Find the usual limit for ball players.

5. Two automobile racers start at the same time and travel in the same direction, their rates being in the ratio 2 : 3. In 5 hours they are 100 miles apart. Find the rate of each.

6. Each of 3 camps had the same number of boys and all were fed from food supplied by the first two. If one of them furnished $\frac{7}{8}$ as much as the other, and the third camp paid the others \$30, how was it divided between the first two camps?

7. The sides of a triangle are 5, 6, and 7 inches, respectively, and in a similar triangle the side corresponding to the shortest side of the first is 10 inches. Find the other sides.

8. The sides of a triangle are 2, 3 and 4 inches, respectively. The perimeter of a similar triangle is 108 inches. Find the lengths of the sides of the second triangle.

Suggestion. The ratio of the perimeters of similar triangles is the same as the ratio of a pair of corresponding sides.

9. The shadow of a tree is 100 feet long. If the shadow of a vertical pole 5 feet in length is 4 feet long, what is the height of the tree?

10. The sides of a triangle are 8, 12, and 15 inches. Find the segments of the longest side made by the bisector of the opposite angle.

11. The sides of a triangle are a, b, and c. Find the segments of each side made by the bisectors of the opposite angles.

12. Use the results of No. 11 to find the segments of the sides of the triangle, which are 16, 18, and 20 inches, made by the bisectors of the opposite angles.

13. If a boy weighing 125 pounds at a distance of 5 feet from the fulcrum is to balance a 75 pound boy, where should the second boy be placed?

Note. If weights are placed on the two ends of a beam and the beam turns about a pivot or fulcrum, the beam will balance when $\frac{w_1}{w_2} = \frac{d_2}{d_1}$. That is, the weights are in inverse ratio to the distances.

14. Where should the fulcrum be placed to balance two boys weighing 120 and 90 pounds respectively, if the beam is 16 feet long?

15. A crowbar 5 feet long is used to lift a stone weighing 780 pounds. The fulcrum of the crowbar is 8 inches from the end of the bar. What force must be applied at the end of the bar to lift the stone?

16. The plan for a building is drawn to a scale of $\frac{1}{4}$ inch to 1 foot. The length of a certain beam is shown on the plan to be $3\frac{1}{4}$ inches. Find the length of the beam.

17. If in a map the distance between two cities 540 miles apart is $2\frac{1}{4}$ inches, what is the distance between two cities which are $3\frac{1}{2}$ inches apart on the map?

18. The perimeter of a triangle is 63 inches and the ratio between the longest and shortest side is $3 : 2$, while the third side is a mean proportional between the other two. Find the sides of the triangle.

19. Wishing to determine the height of a flag staff, a man noticed that by holding a 12 inch ruler vertically in front of his line of sight, at a distance of 2 feet from his eye, the ruler and the flag staff subtended the same angle. How tall was the staff if the man was 200 feet from the base of the staff?

20. The corresponding sides of two similar triangles are 5 inches and 8 inches. If the area of the first is 50 square inches, what is the area of the second?

21. The area of a square is 144 square inches. What is the ratio of a side of this square to the side of another square whose area is 9 times as great?

22. The areas of two similar triangles are 50 and 200 square inches, respectively. If the base of the first is 10 inches, what is the base of the second?

23. The altitude to the hypotenuse of a right triangle is 9 inches. If the entire length of the hypotenuse is 30 inches, find the segments of the hypotenuse.

24. A secant to a circle is 63 inches long. If a tangent to the circle from the same point is 21 inches, find the length of the external part of the secant.

25. A line 8 inches long is divided in extreme and mean ratio. Find the parts of the line.

26. A diameter of a circle is 20 inches in length. A perpendicular is drawn to the diameter from a point on the circle. If the length of the perpendicular is 8 inches, find the segments of the diameter.

Note. The perpendicular to the diameter from a point on the circle is a mean proportional between the segments of the diameter.

27. Two sides of a triangle are 12 and 18 inches, respectively. A line parallel to the base divides the shorter side in the ratio 1 : 2. Find the segments of the longer side.

28. Two corresponding sides of two similar polygons are 5 and 7 inches, respectively. If the area of the smaller polygon is 150 square inches, find the area of the larger.

29. A father divides $3000 among his three sons so that the eldest son receives 20% more than the second, and the second receives 25% more than the youngest. Find the amount each one receives.

30. The difference of the squares of two numbers in the ratio 5 : 2 is 168. Find the numbers.

211. Variation. When is one variable said to be a function of another? (See § 157.)

In the equation $y=x^2$, y is a function of x. By solving the equation for x in terms of y, we get $x= \pm\sqrt{y}$, which expresses x as a function of y.

One variable is said to **vary as** another when the first is equal to the product of the second and a **constant.**

In the equation $x=ky$, x varies as y since x equals the product of y and a constant.

One variable is said to **vary inversely** as another when the first is the quotient of a constant and the second variable.

In the equation $x=\frac{k}{y}$, x varies inversely as y. Evidently, this may also be written $xy=k$.

One variable is said to **vary jointly** as two others when the first variable equals the product of the others and a constant.

In the equation $x=kyz$, x varies jointly as y and z.

212. Many of the formulas of mathematics may be expressed in terms of variation.

For example, $A=\pi r^2$ expresses the fact that the area of a circle varies as the square of the radius.

Similarly, $V=\frac{4}{3}\pi r^3$ expresses the fact that the volume of a sphere varies as the cube of the radius.

Also, $A=\frac{1}{2}bh$ expresses the fact that the area of a triangle varies jointly as its base and altitude.

It is customary to represent the **constant** in each problem of variation by the letter k.

Exercise 202

1. If x varies as y, and $x=14$ when $y=7$, find x when $y=9$.

Solution. Substituting $x=14$ and $y=7$ in $x=ky$, gives $14=7k$, or $k=2$. The reduced equation is $x=2y$, and when $y=9$, $x=18$.

2. If x varies as y and $x=\frac{1}{2}$ when $y=\frac{3}{8}$, find x when $y=5\frac{3}{8}$.

3. In the formula for the volume of the sphere, what is the constant and what are the variables? Find the constant correct to .0001.

4. If x varies inversely as y, and if $x=8$ when $y=3$, find x when $y=\frac{4}{3}$.

5. Write a formula to show that the pressure in pounds of the wind on a sail varies jointly as the area of the sail and the square of the wind's velocity.

6. The distance that a body falls from a state of rest varies as the square of the time it falls. Express this fact by an equation and determine the constant, k, if the body falls the first foot in $\frac{1}{4}$ of a second.

7. The intensity of light varies inversely as the square of the distance of the light from the surface illuminated. If a 20-candle power light 4 feet from the page furnishes a comfortable reading light, how far from the page must a 40-candle power light be placed? A 100-candle power?

8. The weight of an object above the surface of the earth varies inversely as the square of its distance from the center of the earth. An object weighs 125 pounds on the surface of the earth (assume it to be 4000 miles from the center). What will be its weight 1000 miles above the surface? 4000 miles?

9. How far above the surface of the earth must the object in No. 8 be carried to reduce its weight one-half?

10. The safe load, w, of a horizontal beam supported at each end is expressed by the formula $w=\frac{kbd^2}{l}$, where b is the breadth, d the depth, and l the length, or distance between supports. If a 2 by 6 yellow pine joist set horizontally on its edge on supports that are 10 feet apart carries safely a load of 600 pounds, what would be the safe load if its supports were 12 feet apart?

11. Would the joist in No. 10, if placed on its side across a stream 12 feet wide, be safe for a man weighing 150 pounds?

12. What is the safe load for a 2 by 10 joist of the same material as in No. 10 placed on its edge and resting on supports 16 feet apart? Of a 3 by 12 resting on supports 20 feet apart?

13. The time required by a pendulum to make one vibration varies directly as the square root of its length. If a pendulum 100 centimeters long vibrates once per second, find the length of a pendulum that vibrates once in 2 seconds. Once in 4 seconds. Once in $\frac{1}{2}$ a second.

14. What is the time of vibration of a pendulum 50 centimeters long? Of a pendulum 81 centimeters long? Of a pendulum 1000 centimeters long?

CHAPTER XIX

PROGRESSIONS

213. Definitions. A **series** is a succession of related terms whose values are determined according to some law.

A series is **finite** or **infinite** according as the number of its terms is finite or infinite.

The variety of different series is unlimited, as the succeeding paragraphs will show.

Exercise 203

Write three or four additional terms for each of the following series:

1. $1, 2, 3, \ldots\ldots$
2. $1, 3, 5, 7, \ldots\ldots$
3. $8, 11, 14, 17, \ldots\ldots$
4. $2, 4, 8, 16, \ldots\ldots$
5. $1+x+x^2+x^3, \ldots\ldots$
6. $1-x^2+x^4-x^6, \ldots\ldots$
7. $1, 4, 16, \ldots\ldots$
8. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\ldots$
9. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\ldots$
10. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots\ldots$
11. $1, \frac{1}{8}, \frac{1}{27}, \frac{1}{64}, \ldots\ldots$

214. The expression $\sum_{n=1}^{n=5}(n^2+1)$ is taken to mean the sum of the series of five terms obtained by substituting in (n^2+1) the numbers from 1 to 5 inclusive. This is a convenient way for writing in compact form any series. Written as an identity we have,

$$\sum_{n=1}^{n=5}(n^2+1) \equiv 2+5+10+17+26.$$

The expression $n=\infty$ means that n **becomes infinite,** or that its value increases beyond all bounds.

Therefore $\sum\limits_{n=1}^{n=\infty}$(any expression containing n) would represent an infinite series, or one the number of whose terms is unlimited.

For example, $\sum\limits_{n=1}^{n=\infty}\dfrac{n}{n^2+1}=\dfrac{1}{2}+\dfrac{2}{5}+\dfrac{3}{10}+\dfrac{4}{17}\cdots\cdots\cdots\cdots\cdots$

Exercise 204

Expand the following series:

1. $\sum\limits_{n=1}^{n=5}n(n+1)$. Ans. $\sum\limits_{n=1}^{n=5}n(n+1)=2+6+12+20+30.$

2. $\sum\limits_{n=1}^{n=5}(n+2)$. 3. $\sum\limits_{n=1}^{n=4}(5n-2)$. 4. $\sum\limits_{n=1}^{n=6}\dfrac{(n+1)\,(n+2)}{n}$.

5. $\sum\limits_{n=1}^{n=5}\dfrac{n(n-3)}{n+1}$. 6. $\sum\limits_{n=1}^{n=6}\dfrac{n(n+1)^2}{2}$. 7. $\sum\limits_{n=1}^{n=6}\dfrac{n(2n+3)}{n+1}$.

Write the first five terms of each of the following series:

8. $\sum\limits_{n=1}^{n=\infty}\dfrac{1}{n+2}$. Ans. $\sum\limits_{n=1}^{n=\infty}\dfrac{1}{n+2}=\dfrac{1}{3}+\dfrac{1}{4}+\dfrac{1}{5}+\dfrac{1}{6}+\dfrac{1}{7}\cdots\cdots\cdots$

9. $\sum\limits_{n=1}^{n=\infty}\dfrac{n}{2n+3}$. 10. $\sum\limits_{n=1}^{n=\infty}\dfrac{1}{n^2+n}$. 11. $\sum\limits_{n=1}^{n=\infty}\dfrac{1}{2^n}$.

12. $\sum_{n=1}^{n=\infty} (\frac{2}{3})^n$. 13. $\sum_{n=1}^{n=\infty} (n+1)^n$. 14. $\sum_{n=1}^{n=\infty} \frac{n}{(n+1)^n}$.

215. Arithmetic progression. If each term after the first of a series is found by adding a **constant difference** to the preceding term, the series is called an **arithmetic progression.**

The **first term** of an arithmetic progression is represented by a; the **common difference,** by d; the **last term,** by l; and the number of terms, by n. The terms in order are represented by a, $a+d$, $a+2d$, $a+3d$, . . .

What is the 5th term? The 12th term? The 20th term?

From the preceding we obtain the formula,

$l=a+(n-1)d$. (I)

Solve this formula for a, n, and d.

We now have at our disposal four formulas which will enable us to determine any one of the four literal numbers, a, l, n, or d, when the other three are given.

Exercise 205

1. Given $a=3$, $d=4$, $n=8$; find l.
2. Given $d=2$, $a=13$, $l=27$; find n.
3. Given $l=31$, $d=4$, $n=6$; find a.
4. Given $l=48$, $a=8$, $n=9$; find d.
5. Given $a=-3$, $d=2\frac{1}{2}$, $n=10$; find l.
6. Given $a=4$, $d=-5$, $l=-26$; find n.
7. Given $l=14\frac{1}{3}$, $n=9$, $d=2$; find a.
8. Given $a=-5$, $l=-26$, $n=8$; find d.
9. A ball rolling down an inclined plane goes 1 foot the first second, 3 feet the second second, and 5 feet the third second. Find how many feet it will go the eighth second.
10. A body falling freely falls 16 feet the first second, 48 the next, and 80 the next. Find how many feet it will fall the tenth second.

11. There are four numbers in arithmetic progression whose sum is 38. The product of the second and third exceeds the product of the first and fourth by 18. Find the progression.

Suggestion. Let $a-3d$, $a-d$, $a+d$, and $a+3d$ represent the numbers.

12. The sum of the first three terms of an arithmetic progression is 15 and the fifth term exceeds the second by 21. Find the eighth term.

216. Arithmetic means. The first and last terms of a series are called the **extremes** and the remaining terms are called the **means.** The problem of inserting a number of arithmetic means between two given extremes resolves itself into finding d, when a, l, and n are given, and writing the series.

Exercise 206

1. Insert 4 arithmetic means between 3 and 23. Solving, $d=4$. Ans. 3, 7, 11, 15, 19, 23.

Suggestion. $n=6$.

2. Insert 5 arithmetic means between 7 and 43.

3. Insert 4 arithmetic means between -7 and 23.

4. Insert 7 arithmetic means between 2 and 4.

5. Insert 3 arithmetic means between a and b and use the results as formulas to insert 3 arithmetic means between 5 and 21.

6. Find the arithmetic mean between a and b; between 5 and 37; between $3x$ and $2y$.

7. A board is held in place by two nails 15 inches apart. Show how to place 8 more nails between the two at equal intervals.

8. The arithmetic mean of two numbers is 12 and the sum of their squares is 306. Find the numbers.

217. The sum of an arithmetic series. If we represent the last term of an arithmetic series by l, then $l-d$ represents the term before the last, $l-2d$ the term preceding that, etc. Then the sum of any series can be expressed as follows:

$$s=a+(a+d)+(a+2d)\dots +(l-2d)+(l-d)+l. \quad (1)$$

Writing the series in reverse order,

$$s=l+(l-d)+(l-2d)\dots +(a+2d)+(a+d)+a. \quad (2)$$

adding (1) and (2),

$$2s=(a+l)+(a+l)+(a+l)\dots +(a+l)+(a+l)+(a+l),$$

or $2s=n(a+l)$.

Therefore $s=\frac{n}{2}(a+l)$. (II)

If we substitute the value of l in the formula $l=a+(n-1)d$ for l in II, we get $s=\frac{n}{2}[2a+(n-1)d]$. (III)

If any three of the five numbers, a, l, d, n, and s of an arithmetic progression are given, the remaining two can be found by means of the preceding formulas, I, II, and III.

Exercise 207

Use formulas I, II, and III to solve the following:

1. Given $a=2$, $l=23$, $n=8$; find d and s.
2. Given $a=-5$, $l=27$, $d=4$; find n and s.
3. Given $a=8$, $l=-10$, $s=-7$; find n and d.
4. Given $a=-9$, $n=6$, $d=7$; find l and s.
5. Given $a=2\frac{1}{2}$, $n=10$, $s=137\frac{1}{2}$; find l and d.
6. Given $a=5$, $d=3$, $s=75$; find l and n.

Suggestion. Substitute first in III and solve the resulting quadratic for n. Will both roots satisfy the problem?

7. Given $l=-26$, $n=8$, $d=-3$; find a and s.
8. Given $l=6$, $n=9$, $s=30$; find a and d.
9. Given $l=14$, $d=4$, $s=24$; find a and n.
10. Given $n=9$, $d=2$, $s=57$; find a and l.

11. A ball rolling down an inclined plane rolls 3 feet the first second, 9 feet the next and 15 the next. Find how far it will roll in 8 seconds.

12. A body falling freely falls 16 feet the first second, 48 feet the next, and 80 the next. Find how far it will fall in 5 seconds; in 8 seconds; in t seconds.

218. Geometric progressions. A **geometric progression** is a series such that the quotient of any term divided by the preceding term is a constant number. This **constant ratio** is represented by r. The letters a, l, n, and s are used in geometric progressions with the same meaning as in arithmetic progressions. 5, 10, 20, 40, . . . is a geometric progression.

If a, ar, ar^2, ar^3 . . . represent the terms of a geometric progression, what is the 8th term? The 10th term? The nth term? The last question suggests the formula, $l=ar^{n-1}$. (I)

Solve this formula for a; for r. Ans. $r=\sqrt[n-1]{\frac{l}{a}}$.

As in arithmetic progressions, the first and last terms are called the **extremes** and the remaining terms the **means.**

Exercise 208

1. Find the 8th term of the series 3, 6, 12 . . . .
2. Find the 5th term of 5, -15, 45 . . . .
3. Given $a=-2$, $r=4$, $n=5$; find l.
4. Given $l=36$, $n=4$, $r=3$; find a.
5. Given $l=56$, $a=7$, $n=4$; find r.
6. Insert 3 geometric means between 3 and 48.

Suggestion. Show $r=\pm 2$, and write both series.

7. Show that the logarithms of the terms of a geometric series form an arithmetic series.

8. Show that $n=\frac{\log l-\log a}{\log r}+1$.

9. The first two terms of a geometric series are x and y. Find the third term.

10. Find the 18th term of 2, 6, 18, ...

Suggestion. Use logarithms.

11. Find the 20th term of 27, 18, 12, 8, ...

12. If one dollar is placed at compound interest at 6%, it is worth, at the end of the first year, \$1.06; at the end of the second year, \$$(1.06)^2$; and at the end of the third year, \$$(1.06)^3$. Explain. Find by logarithms its value at the end of the 12th year.

13. Find the value of one dollar at compound interest at 5% at the end of 100 years.

14. Find the value of \$100 at compound interest at 6% at the end of 8 years.

219. Sum of a geometric series. The formula for the sum of a geometric series is discovered by the following process;

(1) $s=a+ar+ar^2+ar^3 \quad . \; . \; . \quad ar^{n-1}$.

Multiplying both members of (1) by r,

(2) $rs=ar+ar^2+ar^3 \quad . \; . \; . \quad ar^{n-1}+ar^n$.

Subtracting (1) from (2), $rs-s=ar^n-a$.

Whence $s=\dfrac{ar^n-a}{r-1}\cdot$ (II)

Since $l=ar^{n-1}$, $rl=ar^n$; then by substitution we have,

$s=\dfrac{rl-a}{r-1}\cdot$ (III)

Exercise 209

1. Solve Formula II for a.

2. Solve Formula III for a; for r; for l.

3. Find the sum of eight terms of the series 1, 2, 4, 8, ...

4. Given $a=3$, $r=3$, $n=5$. Find s.

5. Find the sum of 6 terms of 3, $3\sqrt{3}$, 9, ...

6. Find the sum of 7 terms of 2, -4, 8, -16, ...

7. Find the sum of 5 terms of 3, 2, $\frac{4}{3}$, $\frac{8}{9}$, ...

8. The fourth term of a geometric progression is 12 and the sixth term is 27. Find the sum of the first 6 terms.

9. Find the sum of seven terms of $1+x+x^2+x^3$ . . . . Reduce the result to simplest form by dividing the numerator by the denominator.

10. Find the sum of five terms of $1-x+x^2$ . . . . Reduce the result to simplest form.

11. A boy asking for a position is offered one cent for his first day's work, and for each succeeding day double that of the previous day. He works 12 days. Find the total amount he receives.

12. A person writes the same letter to each of three of his friends, and asks them each one to write the same letter to three friends, and so on till the tenth set of letters has been written. Find the total number of letters that will be written in the entire chain.

220. Infinite geometric series. When in any given series r is less than one, it will be observed that each term is less than the preceding term, and as the number of terms becomes greater, the last term becomes smaller and smaller and approaches zero as a limit. In the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8},$ . . . . , it is very easy to find a term that is less than any given amount, as, for example, .001. See § 236.

Find the sum of six terms of this series; eight terms; twelve terms. What do you notice about the sum?

Find by logarithms the 40th term of the series 10, 9, 8.1, ... Find the 100th term.

In the formula $s=\frac{ar^n-a}{r-1}$, when r is less than one and the number of terms is unlimited, it will be observed that ar^n approaches zero as a limit. Therefore for any decreasing infinite series $s \doteq \frac{-a}{r-1}$, or $s \doteq \frac{a}{1-r}$. (Read $\doteq$ "approaches as a limit.")

221. Repeating decimals. A common fraction in its lowest terms, whose denominator contains prime factors other than 2 and 5, cannot be expressed exactly in decimal form. When we attempt to reduce such a fraction to decimals, we find that the same groups of digits occur repeatedly. As, for example,

$\frac{1}{3} = .3333 \ldots$
$\frac{5}{11} = .454545 \ldots$
$\frac{3}{7} = .428571428571 \ldots$

Any number in repeating decimal form may be considered as an infinite geometric series. As, for example,

$$.3333 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots$$

Here $a = \frac{3}{10}$, $r = \frac{1}{10}$. Then $s = \dfrac{\frac{3}{10}}{1 - \frac{1}{10}} = \dfrac{\frac{3}{10}}{\frac{9}{10}} = \frac{1}{3}$.

Exercise 210

Find the sum of each of the following infinite series:

1. $2, 1, \frac{1}{2}, \ldots .$ Ans. 4.
2. $3, 2, \frac{4}{3}, \ldots .$
3. $3, -2, \frac{4}{3}, -\frac{8}{9}, \ldots .$
4. $27, 18, 12, \ldots .$
5. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \ldots .$
6. $6, 3\sqrt{2}, 3, \ldots .$
7. $4, \dfrac{2}{\sqrt{3}-1}, \dfrac{1}{4-2\sqrt{3}}$.
8. $1, x, x^2, \ldots .$ (When x is less than 1.)

Find the values of the following repeating decimals:

9. $.4545 \ldots .$

Suggestion. As a series $.4545 \ldots = \frac{45}{100} + \frac{45}{10000} \cdot \ldots$ Ans. $\frac{5}{11}$.

10. $.6363 \ldots .$
11. $3.16363 \ldots .$

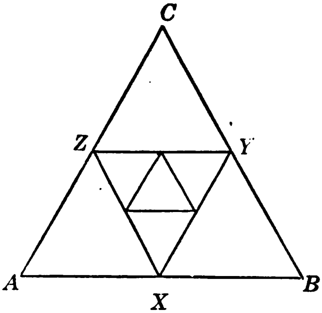
Note. This is 3.1 plus the series $\frac{63}{1000} + \frac{63}{100000} \ldots .$ Ans. $3\frac{9}{55}$.

12. 8.1666 . . . . **13.** 5.0333 . . . .

14. .243243 . . . . **15.** 1.44144144 . . . .

16. If a ball, dropped from a height of 60 feet, rebounds 30 feet on striking the ground, and again rebounds 15 feet, and so on, how far will it travel before coming to rest?

17. One side of the equilateral triangle ABC is 10 inches. The mid-points of the sides are connected forming the triangle XYZ, and so on. If this process is continued indefinitely, find the total length of the lines.



Exercise 211. Review

1. Suppose that every term of an arithmetic progression is multiplied by k; is the result an arithmetic progression?

2. Show that the quotients form a geometric progression when each term of a geometric progression is divided by the same number.

3. What is the sum of the first 200 numbers that are divisible by 5?

4. How many multiples of 7 are there between 350 and 1210?

5. The sum of four numbers of an arithmetic progression is 0, and the sum of their squares is 125. Find the numbers.

6. Show that the sum of $2n+1$ consecutive integers is divisible by $2n+1$.

7. Show that the sum of the arithmetic progression 1, 3, 5, 7, . . . is n^2 where n is the number of terms.

8. Find the sum of all numbers under 200 that are divisible by 3 and not divisible by 2.

9. The product of the 3 terms of a geometric progression is 512. If the first term is 1, find the second.

10. The sum of the first eight terms of a geometric progression is 17 times the sum of the first four. Find r.

11. Find the sum of the first six terms of a geometric progression if the second term is 5 and the fifth, 625.

12. Three numbers form a geometric progression and they are the second, fourth, and ninth terms of an arithmetic progression whose first term is 1. Find d.

13. What is the fourth term of a geometric progression if the second is $\frac{1}{5}$ and the fifth is 625?

14. Find the sum of five consecutive powers of 3 beginning with the first.

15. Of three numbers in geometric progression, the sum of the first and second exceeds the third by 3, and the sum of the first and third exceeds the second by 21. Find the numbers.

16. Three numbers form a geometric progression. If 2 is subtracted from the first, 4 from the second, and 13 from the third, the results form an arithmetic progression, the sum of whose terms is 30. Find the arithmetic progression.

17. To what sum will \$1 amount at 4% compound interest in 5 years?

18. The number of oranges in a pile in the form of a triangular pyramid is $1+(1+2)+(1+2+3)+ \ldots$ depending on the number of layers. How many oranges in a pile of 10 layers?

19. If a, b, and c form a geometric progression, show that $\frac{1}{b-a}$, $\frac{1}{2b}$, and $\frac{1}{b-c}$ form an arithmetic progression.

20. A farmer hires a laborer for the summer at a beginning salary of \$50 a month, with either a raise of \$10 per month,

after the first month, or a raise of $2.50 every two weeks after the first half month. If 4 weeks are considered a month, which is the better proposition for the laborer, provided he will work 20 weeks?

21. A man travels at a constantly increasing rate. He goes 1 mile the first hour, 3 miles the next, 5 the next, and so on. How far will he go in $6\frac{1}{4}$ hours?

22. A well-drilling company in estimating the number of linear feet of casing in stock, find they have 10 piles of 30-foot sections each pile being arranged in triangular form, 12 pipes in the first layer, 11 in the next, etc. How many feet of casing in their stock?

23. In a potato race, the potatoes are placed in a row, the first 10 feet from a box and the remaining ones at intervals of 4 feet. There are 25 potatoes in all. The contestant starts at the box and fetches them one at a time. How far has he run when all the potatoes are placed in the box?

24. Neglecting the resistance of the air a body falls from rest 16.08 feet the first second, 48.24 feet the second second, 80.40 feet the third, etc. How far will it fall in 6 seconds? in 12 seconds?

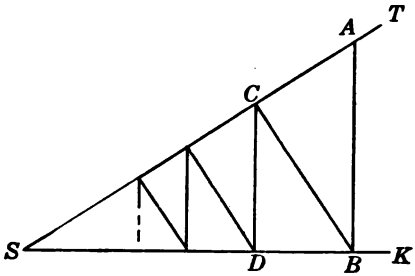
25. The increase in velocity as represented in No. 24 is called acceleration due to gravity and is represented by the literal number g. Using this value for d, how far does the body fall in t seconds? The result is a well-known formula in Physics. Ans. $s=\frac{1}{2}gt^2$.

26. If a falling body is given an initial velocity of v feet per second, then the distance it falls in t seconds is vt feet plus the distance it would fall if starting from rest, i. e., $s=vt+\frac{1}{2}gt^2$.

How long will it take a body to fall 3918 feet, if given an initial velocity of 20 feet per second?

27. What must be the initial velocity of a falling body in order that it shall fall 1200 feet in $7\frac{1}{2}$ seconds?

28. A pulley rolling on a cable which is inclined at an angle of 30°, goes down at the rate of 7.25 feet the first second and in each succeeding second 14.5 feet more than in the preceding one. How long will it take the pulley to reach the end of a 261-foot cable?



29. In the accompanying figure, AB is perpendicular to SK, BC is perpendicular to ST, CD to SK, ... If AB is 12 inches and CB is 10 inches, what is the sum of all the perpendiculars?

30. If an air pump draws out at each stroke 1/10 the volume of the air in the bell jar, what fractional part of the air will remain in the jar at the end of the tenth stroke?

31. There is an Eastern legend that the ruler for whom the game of chess was invented foolishly agreed to pay the inventor 1 grain of wheat for the first square on the board, 2 for the second, 4 for the third, 8 for the fourth, . . .

Determine by logarithms the number of digits in the number of grains of wheat that the inventor should have received and the four figures at the extreme left of this number.

CHAPTER XX

THE BINOMIAL THEOREM

222. The powers of $(a+b)$ and of $(a-b)$.

The following powers are obtained by multiplication:

$(a+b)^1=a+b.$

$(a+b)^2=a^2+2ab+b^2.$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$

$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$

$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$

Similarly, find the corresponding powers of $a-b$.

In the powers of $(a\pm b)^n$, when n is a positive integral number, observe the following:

(1) The number of terms is one greater than n.

(2) The first term is a^n and the exponent of a decreases by unity in each succeeding term.

(3) The exponent of b is unity in the second term and increases by unity in each succeeding term.

(4) The coefficient of the second term is n, and if the coefficient of any term is multiplied by the exponent of a in that term, and the product divided by the number of the term, the quotient is the coefficient of the next term.

(5) The signs of $(a+b)^n$ are all $+$ and the signs of $(a-b)^n$ are alternately $+$ and $-$.

Illustrative examples.

1. Expand $(a+b)^7$.

Solution. The first term is a^7 and the second term is $7a^6b$. The coefficient of the third term is found by multiplying 7 by 6 and dividing the product by 2. The complete third term is $21a^5b^2$.

Continue in the same way to find succeeding terms. The expansion is,

$$(a+b)^7=a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7.$$

2. Expand $(2x^2-3y^3)^3$.

Solution. $(a-b)^3=a^3-3a^2b+3ab^2-b^3$.

Using this as a formula, we get,

$$(2x^2-3y^3)^3=(2x^2)^3-3(2x^2)^2(3y^3)+3(2x^2)(3y^3)^2-(3y^3)^3=$$
$$8x^6-36x^4y^3+54x^2y^6-27y^9.$$

Exercise 212

Expand the following:

1. $(a-b)^6$. **2.** $(x+y)^7$. **3.** $(x-y)^8$. **4.** $(2x+3y)^4$.

5. $(3a-2b)^5$. **6.** $(2n^2+3m^2)^5$. **7.** $(3x^2-5y)^4$.

8. $(7R^2-6r)^3$. **9.** $(3m^2n-4)^5$. **10.** $(6xy^3-2a^2b)^4$.

11. $\left(2x+\frac{y}{2}\right)^4$. **12.** $\left(\frac{x}{y}+\frac{m}{n}\right)^3$. **13.** $\left(\frac{x^2}{y^2}-\frac{y^2}{x^2}\right)^4$.

223. Any required term. If the coefficients of the terms of the binomial expansion are determined by (4) of Section 222, and the coefficients are left in the fractional form, a simple process for writing any term is discovered. For example,

$$(a\pm b)^6=a^6\pm\frac{6}{1}a^5b+\frac{6\cdot5}{1\cdot2}a^4b^2\pm\frac{6\cdot5\cdot4}{1\cdot2\cdot3}a^3b^3+$$
$$\frac{6\cdot5\cdot4\cdot3}{1\cdot2\cdot3\cdot4}a^2b^4\pm\frac{6\cdot5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4\cdot5}ab^5+\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4\cdot5\cdot6}b^6.$$

Similarly, write the expansion of $(a\pm b)^7$.

Observations.

(1) The factors of each numerator are $n\ (n-1)\ (n-2)$. . .

(2) The factors of each denominator are $1\cdot2\cdot3$ . . .

(3) The number of factors in each numerator and denominator is one less than the number of the term.

(4) The exponent of b is one less than the number of the term and the exponent of a is found by subtracting the exponent of b from n.

(5) The only signs that are negative are the signs of the even numbered terms of $(a-b)^n$.

Note. These observations may all be condensed into the **formula:**

The rth term of $(a\pm b)^n\equiv$

$$\frac{n(n-1)\ (n-2)\ .\ .\ .\ \text{to}\ (r-1)\ \text{factors}}{1\cdot2\cdot3\ .\ .\ .\ \text{to}\ (r-1)\ \text{factors}}a^{n-r+1}b^{r-1}.$$

Illustrative examples.

1. Find the 5th term of $(x+y)^8$.

Solution. 5th term of $(x+y)^8=\frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4}x^4y^4=70x^4y^4$.

2. Find the 6th term of $(2n-3m)^9$.

Solution. 6th term of $(2n-3m)^9=-\frac{9\cdot8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4\cdot5}(2n)^4(3m)^5$

$=-489888n^4m^5$. (Explain the negative sign.)

Exercise 213

1. Find the 7th term of $(a+b)^{11}$.
2. Find the 8th term of $(x-y)^{12}$.
3. Find the 5th term of $(2a-b)^9$.
4. Find the 4th term of $(3n+2m)^7$.
5. Find the 3rd term of $(2a-\sqrt{b})^6$.
6. Find the 6th term of $\left(3a^2-\frac{b}{2}\right)^7$.
7. Find the 4th term of $\left(\frac{x}{2}-\frac{y}{3}\right)^6$.

It is explained in the work of the preceding paragraphs that the following formula holds true for the expansion of $(a\pm b)^n$, when n is a positive integral number.

$$(a\pm b)^n\equiv a^n\pm\frac{n}{1}a^{n-1}b+\frac{n(n-1)}{1\cdot2}a^{n-2}b^2\pm$$
$$\frac{n(n-1)(n-2)}{1\cdot2\cdot3}a^{n-3}b^3\ \dots\ .$$

It will be noticed that the number of terms is one more than n and, if one should attempt to find more than $n+1$ terms by the formula, as the 5th term of $(a+b)^3$, one of the factors of the numerator of the coefficient is zero which makes the term zero.

The series of terms in the right member of the above is called the expansion of $(a\pm b)^n$. The series is finite and contains $n+1$ terms only when n is a positive whole number. The whole identity is called the **binomial theorem.**

224. Binomial theorem, exponent fractional or negative. It is proved in higher mathematics that the binomial theorem is true for fractional and negative exponents when the first term of the binomial is arithmetically greater than the second. The expansion of any binomial with a fractional or negative exponent gives an infinite series, which can be interpreted if the larger term, arithmetically, of the binomial is the first term.

Illustrative examples.

1. Expand $(1-x)^{-1}$.

Solution. Substituting in the formula,

$$(1-x)^{-1}=1-(-1)x+\frac{(-1)(-2)}{1\cdot 2}x^2-\frac{(-1)(-2)(-3)}{1\cdot 2\cdot 3}x^3+\ \ldots$$
$$=1+x+x^2+x^3+\ \ldots\ .$$

Can you write additional terms by inspection?

It will be noticed that the expansion of $(1-x)^{-1}$ is an infinite geometrical progression with the first term 1 and the ratio x. This series is of meaning to us only when x is less than 1. Applying the formula for finding the sum of the geometric series $1+x+x^2+x^3\ \ldots$ we get $\frac{1}{1-x}$ which is another way of writing $(1-x)^{-1}$.

2. Expand $(1+x)^{\frac{1}{2}}$.

Solution. $$(1+x)^{\frac{1}{2}}=1+\tfrac{1}{2}x+\frac{\frac{1}{2}(-\frac{1}{2})}{1\cdot 2}x^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{1\cdot 2\cdot 3}x^3\ \ \ldots$$
$$=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}\ \ \ldots$$

3. Write the rth term of $(1+x)^{\frac{1}{2}}$.

Solution. By a careful study of Example 2, it will be observed that the factors of the numerator of the rth term of $(1+x)^{\frac{1}{2}}$ are $(\frac{1}{2})\ (-\frac{1}{2})\ (-\frac{3}{2})\ (-\frac{5}{2})\ \ldots$ to $r-1$ factors, while the factors of the denominator are $1\cdot 2\cdot 3\ \ldots$ to $r-1$ factors. If each factor of the numerator and denominator is multiplied by 2, the rth term reduces to

$$\frac{(1)(-1)(-3)(-5)\ \ldots\ r-1 \text{ factors}}{2\cdot 4\cdot 6\cdot 8\ \ldots\ r-1\text{ factors}}\,x^{r-1}.$$

Exercise 214

1. Expand $(a-b)^8$ to 4 terms.

2. Expand to 4 terms $(1+x)^{-1}$. Check by performing the division $\frac{1}{1+x}$.

3. Expand to 4 terms $(1-x)^{-2}$. Check by performing the division $\frac{1}{1-2x+x^2}$.

4. Expand to 4 terms $(1+x)^{\frac{2}{3}}$.

5. Expand to 4 terms $(1-2x)^{\frac{3}{4}}$.

6. Expand to 4 terms $(8x^3-2y)^{\frac{4}{3}}$.

7. Expand to 3 terms $(1+x)^{-\frac{1}{2}}$.

8. Expand to 4 terms $(x-2y)^{-3}$.

9. Expand to 4 terms $(4x^2-5)^{\frac{1}{4}}$.

10. Expand to 4 terms $(2x+y)^{10}$.

11. Expand to 4 terms $(5x^2+3b)^5$.

12. Find the 6th term of $(x-2y)^{10}$.

13. Find the 7th term of $(3x-2y)^{11}$.

14. Find the 6th term of $(1-2x)^{-1}$.

15. Find the 5th term of $(1+x)^{\frac{1}{2}}$.

16. Find the 8th term of $(2-x)^{-\frac{1}{2}}$.

17. Expand to 3 terms $(1+2x)^{\frac{1}{2}}$ and check by finding the square root of $1+2x$.

225. Square root and cube root. The binomial theorem and its applications furnish us with the solution of many practical problems. One of these is the extraction of any root of any number. Square root and cube root only will be treated here. As a preliminary step, we expand the following:

$$(1\pm x)^{\frac{1}{2}}=1\pm\tfrac{1}{2}x-\tfrac{1}{8}x^2\pm\tfrac{1}{16}x^3- \quad .\ .\ .\ . \qquad (1)$$

$$(1\pm x)^{\frac{1}{3}}=1\pm\frac{x}{3}-\frac{1}{9}x^2\pm\frac{5}{81}x^3- \quad .\ .\ .\ . \qquad (2)$$

Illustrative examples.

1. Find the square root of 37.

Solution. Substituting in (1),

$$\sqrt{37}=(36+1)^{\frac{1}{2}}=6(1+\tfrac{1}{36})^{\frac{1}{2}}$$
$$=6\left(1+\frac{1}{72}-\frac{1}{8\cdot 36^2}+\frac{1}{16\cdot 36^3}\quad\cdot\;\cdot\;\cdot\right)$$
$$=6+.08333-.00058+.000008=6.083-.$$

2. Find the cube root of 76.

Solution. Using (2),

$$\sqrt[3]{76}=(64+12)^{\frac{1}{3}}=4\left(1+\frac{3}{16}\right)^{\frac{1}{3}}=4\left(1+\frac{1}{16}-\frac{1}{256}+\frac{5}{12288}\right.$$
$$\left.\cdot\;\cdot\;\cdot\quad\right)=4+.25-.015625+.0016=4.236-.$$

Exercise 215

Find the square root of the following numbers, using four terms of (1):

1. 26. **2.** 38. **3.** 53. **4.** 150. **5.** 87. **6.** 200.

Find the cube root of the following numbers, using four terms of (2):

7. 28. **8.** 66. **9.** 130. **10.** 231.

CHAPTER XXI

CUMULATIVE REVIEW

226. Factoring and fractions.

Exercise 216

Factor the following:

1. $x^3+3x^2-4x-12.$
2. $(x+2)^2(x-5)-(x-3)(x-5)-5(x-5).$
3. $a^2-25b^2-6a+9.$
4. $2(a^3+1)-7(a^2-1).$
5. $a^4+a^2b^2+b^4.$
6. $x^4+2x^2+9.$

Simplify the following:

7. $\dfrac{8a^3+b^3}{9a^2-4b^2}\cdot\left(1+\dfrac{4b}{3a-2b}\right)\div\dfrac{2a+b}{9a^2-12ab+4b^2}.$

8. $\left[\dfrac{a^4+a^2+1}{a^3+1}\cdot\dfrac{(a+1)^2}{a^2-1}\div\dfrac{a^3-1}{(a-1)^2}\right]-1.$

9. $\dfrac{x+2a}{x-a}-\dfrac{x^2+7ax}{a^2-x^2}+\dfrac{x-2a}{x+a}.$

10. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{1}{n^2}-\dfrac{1}{m^2}}\div\dfrac{a^2-b^2}{m^3-n^3}.$

11. $\dfrac{\dfrac{x+y}{y}+\dfrac{y}{x+y}}{\dfrac{1}{x}+\dfrac{1}{y}}.$

12. $a+\dfrac{a}{a+\dfrac{1}{a}}=a+\dfrac{a}{\dfrac{a^2+1}{a}}=a+\dfrac{a^2}{a^2+1}=?$

13. $x-\dfrac{x}{x-\dfrac{x}{x-\dfrac{1}{x}}}=x-\dfrac{x}{x-\dfrac{x}{\dfrac{x^2-1}{x}}}=x-\dfrac{x}{x-\dfrac{x^2}{x^2-1}}=?$

14. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{2}}}}$. Ans. $\dfrac{5}{8}$.

15. $\dfrac{\dfrac{3}{4}\cdot\dfrac{3}{4}-1}{\dfrac{3}{4}\cdot\dfrac{3}{4}\cdot\dfrac{3}{4}-1}$.

16. $1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}}$.

17. $\dfrac{a-2}{a-2-\dfrac{a}{a-\dfrac{a-1}{a-2}}}$.

18. $\dfrac{1}{m-\dfrac{m^2-1}{m+\dfrac{1}{m-1}}}$.

227. Exponents and radicals.

Exercise 217

Find the numerical value of each of the following:

1. $(-64)^{-\frac{2}{3}}$.
2. $16^{-\frac{3}{4}}\div 4^{\frac{3}{2}}$.
3. $9^{\frac{3}{2}}\div 27^{-\frac{2}{3}}$.
4. $\dfrac{16^{-\frac{3}{4}}\cdot 27^{\frac{4}{3}}}{9^{\frac{3}{2}}\cdot 64^{-\frac{2}{3}}}$.
5. $3^{-2}\cdot\dfrac{7}{12^{-1}}$.
6. $(6^{-2}\cdot 4^{-2}\cdot 3^3)\div(2^{-8}\cdot 9^{-2})$.
7. $8^0\cdot 9^{-1}\cdot 3^{-2}\div 27^{-1}$.

Solve for a:

8. $27^a\cdot 9=3^{2a}$.
9. $8^{2a}\cdot 4^{3a}=16^3$.
10. $(9^a)^{2a}=\dfrac{3}{27^{-a}}$.
11. $36^{2a}=\dfrac{(6^a)^a}{6^{-4}}$.
12. $a^{-\frac{2}{3}}=8$.
13. $a^{-\frac{3}{4}}=125$.
14. $\dfrac{\sqrt[3]{a^{\frac{4}{3}}}}{\sqrt[3]{a^{\frac{1}{3}}}}=\dfrac{\sqrt[5]{5}}{\sqrt[5]{6}}$.
15. $\dfrac{1}{8}\,a^{-\frac{4}{5}}=2$.

Simplify:

16. $(x^{a-1})\,(x^{2-a})^2\,(x^{a-4})^{-1}$.
17. $\left(x^{\frac{a+2}{a+3}}\right)\left(x^{\frac{1}{a+3}}\right)$.
18. $x^{a^3-b^3}\div x^{a-b}$.
19. $\left(x^{\frac{1}{n^2-1}}\right)^{n+1}$.

20. $\left(a^{\frac{1}{2}}x^{-\frac{1}{2}}\sqrt{ax^{-\frac{1}{4}}\sqrt[3]{x}}\right)^{\frac{1}{2}}$.

21. $\left(\sqrt{x^{a^2}} \div \sqrt[3]{x}\right)^{\frac{3}{3a-1}}$.

22. $\dfrac{a\sqrt[4]{x^3}}{\sqrt{y}\sqrt{a}} \div \dfrac{\sqrt{a^{-2}}}{\sqrt[3]{ax}\cdot x^{-\frac{1}{6}}}$.

23. $\left(m^{-\frac{1}{2}}x^{\frac{1}{3}}\sqrt{mx^{-\frac{1}{3}}\sqrt[4]{x^{\frac{4}{3}}}}\right)^{\frac{1}{3}}$.

24. $\left(\sqrt{a^2b^{-1}c\sqrt{b^2c^{-1}a\sqrt{c^2a^{-1}b\sqrt{abc}}}}\right)^{16}$.

25. $x^{\frac{a}{a+b}} \div x^{\frac{b}{a+b}}$.

26. $[\{(x^a)^{-b}\}^{-c}] \div [\{(x^c)^{-a}\}^{b}]$.

27. $(ab)^{x+y} \div a^x b^y$.

28. $[(a^{x+y})^{x-y} \div (a^{y-x})^{x}]$.

29. $\left(x^{\frac{a}{4}}y^{-\frac{c}{3}}z^{ac}\right)^{\frac{12}{ac}}$.

30. $(4^{a+2}+4\cdot 4^a) \div (16\cdot 4^{a+2})$.

31. $\sqrt{\dfrac{3^{n+2}}{9^{-n}} \div \dfrac{27^n}{3^3}}$.

32. $(a^{x-1})^3(a^{2+x})^3(a^{x-3})^{-2}$.

33. $\dfrac{5^{-2}-3^{-2}}{5^{-3}-3^{-2}}$.

34. $\dfrac{x^{-1}+3a^{-1}}{x^{-3}+27a^{-3}}$.

35. $\dfrac{a^{-4}+a^{-2}+1}{a^{-2}-a^{-1}+1}$.

36. $\dfrac{(.008)^{-\frac{1}{3}}\cdot\sqrt{25}}{(.04)^{-\frac{3}{2}}\cdot(2.25)^{-\frac{1}{2}}}$.

37. $\dfrac{1-x^{-2}b^2}{x^{-1}-x^{-2}b}\cdot\dfrac{x^{-1}b^{-1}}{x^{-1}+b^{-1}}$.

38. $\dfrac{8\cdot 2^3\cdot 4^{n-5}}{4\cdot 2^{2n}\cdot 8^2}$.

39. $(a^{5x}-5a^{3x}+10a^{x}-10a^{-x}+5a^{-3x}-a^{-5x})$
$\div(a^{2x}+a^{-2x}-2)$.

40. $(5\sqrt{5}-\sqrt{7}+9\sqrt{3}+2\sqrt{105})(\sqrt{3}-\sqrt{7}+\sqrt{5})$.

41. $(3\sqrt{\tfrac{2}{3}}-2\sqrt{\tfrac{1}{3}}+10\sqrt{\tfrac{1}{5}})(\tfrac{1}{2}\sqrt{24}+\tfrac{1}{5}\sqrt{125}+\sqrt{108})$.

42. $(\sqrt{a+1}-\sqrt{a-1})(\sqrt{a-1})$.

43. $(\sqrt{a+1}-2)^3$.

Rationalize the denominators:

44. $\dfrac{\sqrt{5}+\sqrt{2}}{(2\sqrt{5}+\sqrt{2})(18+4\sqrt{10})}$.

45. $\dfrac{\sqrt{3-\sqrt{2}}}{\sqrt{2-\sqrt{3}}\sqrt{3+\sqrt{2}}}$.

46. $\dfrac{\sqrt{7}-\sqrt{3}}{2+\sqrt{3}-\sqrt{7}}$.

47. $\dfrac{1}{2+\sqrt{3}+\sqrt{5}+\sqrt{2}}$.

48. $\dfrac{3+\sqrt{2}}{\sqrt{3}+\sqrt{3+\sqrt{2}}}$.

49. $\dfrac{5\sqrt{3}+3\sqrt{2}}{\sqrt{2}+\sqrt{3}}$.

50. $\dfrac{\sqrt{26+8\sqrt{3}}}{\sqrt{6}-\sqrt{2}}$.

51. $\dfrac{2ab}{\sqrt[5]{9a^3b^2}}$.

52. $8\sqrt{16\frac{1}{3}}+2\sqrt{27}-5\sqrt{363}+6\sqrt{\dfrac{100}{3}}$.

53. $\dfrac{1}{1-\sqrt{-2}}$.

54. $\dfrac{2+\sqrt{-2}}{2-\sqrt{-2}}$.

55. $\dfrac{\sqrt{a^3}}{\sqrt{-a^5}}$.

56. $\dfrac{\sqrt{3}+\sqrt{-2}}{1-\sqrt{-1}}$.

57. $\dfrac{\sqrt{-3}-\sqrt{-2}}{\sqrt{-3}+\sqrt{-2}}$.

58. $\dfrac{x-\sqrt{y-z}}{x+\sqrt{z-y}}$.

59. $\dfrac{4\sqrt{-2}+3\sqrt{2}}{2\sqrt{-2}-3\sqrt{2}}$.

60. $\dfrac{3m-2n\sqrt{-1}}{2m-n\sqrt{-1}}$.

Extract the square root of:

61. $a^{-4x}+4a^{-3x}-2a^{-2x}-12a^{-x}+9$.

62. $4a^3+5a^2-11a+4-12a^{\frac{5}{2}}+14a^{\frac{3}{2}}-4a^{\frac{1}{2}}$.

63. $a^5-2a^4x^{-1}+5a^3x^{-2}-6a^2x^{-3}+6ax^{-4}-4x^{-5}+a^{-1}x^{-6}$.

64. $a^6-\dfrac{a^5}{2}+\dfrac{33a^4}{16}-\dfrac{a^3}{2}-\dfrac{a^3}{b}+a^2+\dfrac{a^2}{4b}-\dfrac{a}{b}+\dfrac{1}{4b^2}$.

65. Find to three decimals: $\sqrt{\dfrac{\sqrt{3}+\sqrt{17}}{\sqrt{120}}}$.

66. Find to two decimals: $\sqrt[3]{\sqrt{10}-\sqrt{5}}$.

67. Collect $\dfrac{1}{(3-\sqrt{2})^2}+\dfrac{1}{(3+\sqrt{2})^2}$.

68. Collect $\dfrac{1}{a-\sqrt{a^2-4}}+\dfrac{1}{a+\sqrt{a^2-4}}$.

69. Simplify $\left(\sqrt{\dfrac{a}{b}+\dfrac{b}{a}}-\sqrt{\dfrac{a}{b}-\dfrac{b}{a}}\right)^2$.

70. Collect $\dfrac{\sqrt{a}+\sqrt{b}}{2\sqrt{b}}-\dfrac{\sqrt{a}-\sqrt{b}}{2\sqrt{a}}$.

228. Equations.

Exercise 218

Solve the following equations:

1. $(2a-1)^2-2(a+2)(2a-7)=19-3a.$
2. $\frac{n}{9}(n-3)-\left(\frac{n+1}{3}\right)^2=\frac{2}{3}(9+n).$
3. $\frac{x}{1-2x}+\frac{x+1}{4x^2-1}=\frac{x-2}{1+2x}.$
4. $\frac{5a^2-2a-4}{a^2+2a-3}+\frac{2a+3}{1-a}=\frac{3a-1}{a+3}.$
5. $\left(\frac{x}{2x+1}-1\right)\left(\frac{x}{x+1}+1\right)+\frac{3x-2}{2x-1}=\frac{3}{7}.$
6. $\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)=6.$
7. $(3a^2-5a+1)^2-5(3a^2-5a+1)+6=0.$
8. $\sqrt{x+5}-\sqrt{x}=1.$
9. $\sqrt{x+4}+\sqrt{2x-1}=\sqrt{7x+1}.$
10. $\sqrt{2x+3}+\sqrt{x+1}=\sqrt{7x+4}.$
11. $\sqrt{2x+3}-\sqrt{x+1}=\sqrt{7x+4}.$
12. $\sqrt{7+2x}+\sqrt{1-x}=\sqrt{2x+15}.$
13. $\sqrt{2x-3}+\sqrt{x-2}-\sqrt{3x-5}=0.$
14. $\sqrt{2x-3}-\sqrt{x-2}-\sqrt{3x-5}=0.$
15. $-\sqrt{2x-3}+\sqrt{x-2}-\sqrt{3x-5}=0.$
16. $\sqrt{2x-3}+\sqrt{x-2}+\sqrt{3x-5}=0.$
17. $\sqrt{3x-2}+\sqrt{2-x}=\sqrt{2x}.$
18. Write and solve three other examples involving the same radicals as No. 17, but differing in signs. Compare with Nos. 13–16.
19. $\sqrt{3x-5}+\sqrt{2x-3}=\sqrt{5x-8}.$
20. Write and solve three other examples involving the same radicals as No. 19.

21. $\sqrt{x-\sqrt{x^2-3\sqrt{x+4}}}=1.$

22. $\sqrt{x+\sqrt{x^2+3\sqrt{x+4}}}=1.$

23. $\sqrt{5+x}-\dfrac{2}{\sqrt{5-x}}=\sqrt{5-x}.$

24. $\dfrac{\sqrt{3+x}}{\sqrt{3-2x}}-\dfrac{\sqrt{3-2x}}{\sqrt{3+x}}=\dfrac{3}{2}.$

25. $x^2-3x+4-2\sqrt{x^2-3x+5}=2.$

26. $x+x^{\frac{1}{2}}=30.$

27. $3x^{-1}-5x^{-\frac{1}{2}}=2.$

28. $5a^{\frac{2}{3}}-3a^{\frac{1}{3}}=14.$

29. $2a^{-\frac{2}{3}}-5a^{-\frac{1}{3}}+2=0.$

Solve the following for x:

30. $2a^2x^2+abx=3b^2.$

31. $abx^2-bx+2ax-2=0.$

32. $anx^2-2ax+3nx=6.$

33. $mx^2-m^2x+mnx-x+m-n=0.$

34. $\left(ax-\dfrac{6}{ax}\right)^2-6\left(ax-\dfrac{6}{ax}\right)+5=0.$

229. Miscellaneous applications.

Exercise 219

1. Given the formulas $i=prt$ and $a=p+prt$, eliminate p and derive a formula for i.

2. Solve $F=32+\dfrac{9C}{5}$ for C.

3. Given the formulas $S=4\pi r^2$ and $V=\frac{4}{3}\pi r^3$, eliminate r and solve for V.

4. Eliminate n from $l=a+(n-1)d$ by the use of $s=\dfrac{n}{2}(a+l)$.

5. Solve $l=ar^{n-1}$ for n in terms of a logarithmic formula.

6. Eliminate a from $s=\dfrac{a-ar^n}{1-r}$ by the use of $l=ar^{n-1}$.

7. Solve the formula $s=\frac{1}{2}gt^2+vt$ for t.

8. Eliminate t from $s=\frac{1}{2}gt^2$ and $v=gt$, and obtain a formula for v in terms of the other letters.

9. Find the roots of $.23x^2-1.21x=2.7$ correct to .001.

10. Find, without logarithms, the value of $(13.7)^{\frac{3}{2}}$.

11. Find, by the use of logarithms, the value of $(.00327)^{-\frac{1}{5}}$.

12. Solve by logs $\sqrt[5]{\dfrac{(2.87^2)\ (.03275)}{1.83^3}}$.

13. Reduce $a^{15}-b^{15}$ to its four prime factors.

14. Eliminate y from $x^2+y=7$ and $x+y^2=11$ and find one root of the resulting equation of the fourth degree by synthetic division. Find the corresponding value of y and check.

15. Solve for x and y:

$$\frac{x-y}{3}\div\left(\frac{x+1}{2}-\frac{y-1}{3}\right)=\frac{2}{3}$$

$$\frac{x-3y}{2}-\frac{2x+y}{3}=8.$$

16. Find four consecutive odd numbers such that the product of the third and fourth exceeds the sum of the squares of the first and second by 13.

17. A has $2.95 in nickels, dimes, and quarters, 20 coins in all. If the number of nickels is one-third the number of dimes and quarters together, find the number of each.

18. A motor-boat requires 6 hours to go 18 miles down stream and return. If the current were one-half as swift, the motor-boat could make the round trip in 4 hours and 48 minutes. Find the rate of the boat and the rate of the current.

19. If a number composed of two digits is multiplied by the digit in units' place, the product is 24 times the sum of the digits. The digit in units' place is 3 more than the digit in tens' place. Find the number.

20. A farmer has in one bin feed composed of 2 parts corn to 3 parts wheat. In another bin he has ground feed com-

posed of 5 parts corn to 3 parts wheat. He wished to obtain 90 pounds of feed half corn and half wheat. How much must he take from each bin?

21. A mechanic determines that if his wages are increased $12\frac{1}{2}$c per hour, it will require 40 hours less time for him to earn \$100. What are his present wages?

22. A locomotive engineer whistled for a crossing at a certain distance from it. The sound of the whistle is heard at the crossing 25 seconds before the arrival of the train. If the speed of the train was 80 feet per second and sound travels 1080 feet per second, how far was the train from the crossing when the engineer blew his whistle?

23. Separate 21 into two parts so that one part increased by 50% of itself is to the square of the other part as $2:9$.

24. Two autos start toward each other from two towns at the same time and meet in 4 hours. One auto travels at a uniform rate of 6 miles per hour faster than the other and requires $3\frac{1}{3}$ hours less time to go the entire distance between the towns. Find the rate of each auto and the distance of one town from the other.

25. A man has two investments, one yielding 3% and the other 4%, from which he derives an annual income of \$590. If the investments were interchanged, his annual income would be \$600. Find the amount of each investment.

26. The arithmetic mean between two numbers is $37\frac{1}{2}$ and their geometric mean is 36. Find the numbers.

27. A and B together can do a piece of work in half the time required by C to do it alone. A works twice as fast as B. Also B and C can do the work together in $4\frac{4}{5}$ days. In what time can each do the work alone?

28. The ages of A and B are 20 and 13 years, respectively. In how many years will their ages have the ratio $4:3$?

29. The denominator of a certain fraction exceeds the numerator by 3. If a certain number is added to both terms

of the fraction, the value of the fraction is $\frac{5}{6}$. If the same number is subtracted from both terms of the fraction, the value is $\frac{7}{10}$. Find the fraction.

30. It is proved in geometry that one side of a regular decagon inscribed in a circle is a mean proportional between the radius and the difference of the radius and the side. Find the side of a regular decagon inscribed in a circle whose radius is 12 inches.

31. Find the apothem and area of the decagon of No. 30.

32. Solve $d=16t^2$ for t and use the resulting formula to determine the number of seconds required for a body to fall 900 feet.

33. A stone is dropped from the top of Washington Monument, the height of which is 555 feet. In how many seconds will it strike the ground?

34. A stone is dropped from the top of a cliff overhanging a river and 5 seconds later its splash is heard in the water below. If sound travels 1080 feet per second, find the height of the cliff. Ans. 350 feet.

35. An auto starts from a town at 8 A. M., traveling at the rate of 7 miles the first hour, 8 miles the second hour, 9 miles the third hour, and so on. A second auto starts at 9 A. M. and travels at a uniform rate of 12 miles per hour. At what times will the two autos be together?

36. A merchant sells an article for $75 and, computing his percentage of profit on the purchase price, finds that his percentage of profit equals the number of dollars in the purchase price. What was the purchase price?

37. A merchant sells an article for $90 and, computing his percentage of profit on the purchase price, finds that his percentage of profit is 10 less than the number of dollars in the purchase price. Find the purchase price.

38. The perimeter of a right triangle is 48 feet and its area is 96 square feet. Find its legs and hypotenuse.

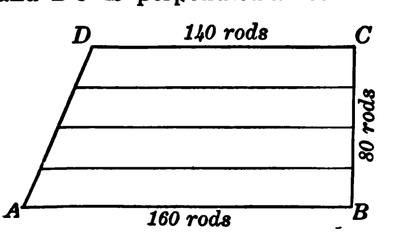
39. The front wheels of a farm wagon make 88 more revolutions per mile than the rear wheels. The front wheels of a farm truck are 2 feet less in circumference and the rear wheels 3 feet less than the corresponding wheels of the wagon, and the front wheels of the truck also make 88 more revolutions per mile than its rear wheels. Find the circumferences of both sets of wheels.

40. It has been shown that π is four times the fraction

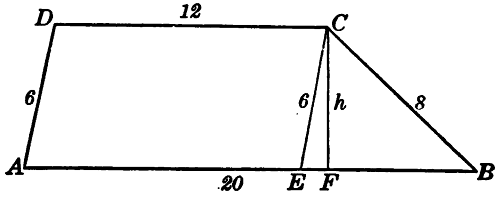
$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{16}{2+\cfrac{25}{2+\cdots}}}}}.$$

Now the value of π, correct to 6 decimals, is 3.141592. How much does four times the value of the fraction, stopping at $2+\frac{64}{2\ \ldots}$, differ from the value of π? How much, stopping at $2+\frac{100}{2\ \ldots}$?

41. A farmer has a field in the shape of the trapezoid of the accompanying figure. Its dimensions are $AB=160$ rods, $BC=80$ rods, and $CD=140$ rods. AB and CD are parallel and BC is perpendicular to both. He wishes to divide it equally among his four sons by fences parallel to AB and CD. How far apart will these fences be, and what frontage will each son have on the road AD?



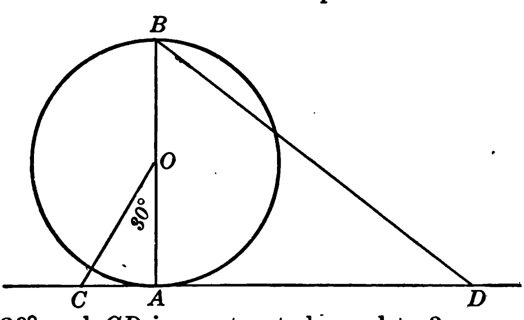
42. Find the area of a triangle whose sides are 12, 16, and 24 inches, respectively.



The area of a trapezoid all of whose sides are known as in the accompanying figure, can be found if h, the distance between the parallel sides, is considered as the altitude of the triangle BCE. The value of h can be found by solving the equations: $x^2+h^2=36$, and $(8-x)^2+h^2=64$, where $x=EF$. Explain.

43. Find the area of the trapezoid whose parallel sides are 24 and 36 feet and whose non-parallel sides are 8 and 10 feet, respectively.

44. The value of the continued fraction $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}$ is one root of $\frac{1}{x}=\frac{x}{1-x}$. Solve the equation, simplify the continued fraction and compare the results.



45. In the accompanying figure, CD is tangent to the circle whose center is O at A, one end of the diameter AB. $\angle AOC=30°$ and CD is constructed equal to $3r$.

Show that $CA=\frac{1}{3}r\sqrt{3}$. Then $AD=3r-\frac{1}{3}r\sqrt{3}$.

Compare the length of BD with the length of the semicircle.

CHAPTER XXII

SUPPLEMENTARY TOPICS

INTERPRETATION OF IMAGINARIES

230. Conjugate imaginaries. Complex numbers that differ only in the sign of the imaginary term are called **conjugate imaginaries.** Thus $5+2\sqrt{-1}$ and $5-2\sqrt{-1}$ are conjugate. The typeforms are $a+bi$ and $a-bi$.

Now $(a+bi)+(a-bi)=2a$, $(a+bi)-(a-bi)=2bi$, and $(a+bi)\ (a-bi)=a^2+b^2$. Evidently the sum and the product of a pair of conjugate imaginaries are real numbers, but the difference is an imaginary number.

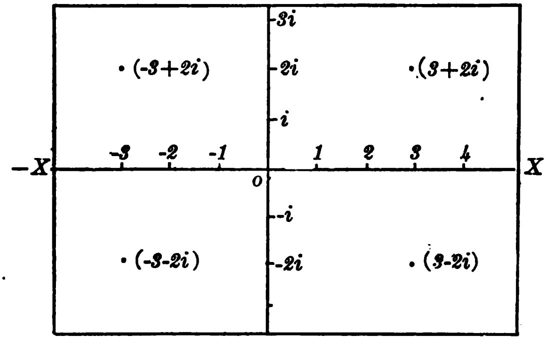
If a complex number is one root of an equation of second or higher degree, then its conjugate is a second root.

Given two complex numbers not conjugate, evidently their sum, difference, product, and quotient are complex numbers unless the imaginary term becomes zero.

Since the square of a complex number is a complex number, the square root of any complex number may be expressed as a complex number following the plan of § 179, except that the rational part is the difference of two factors of m.

231. Graph of a complex number. Since the symbol $\sqrt{-1}$, or i, can be interpreted as an operator that turns a real number through an angle of 90° (see § 180), it may be used to indicate direction on a plane as the signs $+$ and $-$ are used to indicate direction on a line. For $+3$ is interpreted to mean a distance of 3 units to the right of an agreed 0 point on a line in contrast with -3 which is interpreted as 3 units to the left of the point. Then $3i$ may mean a distance of 3 units measured upward on a line which is perpendicular to the first line at the point O, and $-3i$ a distance downward on the same perpendicular line.

A complex number may be represented graphically as a point on a plane; for $3+2i$ is to be found by moving 3 units to the right of O then 2 units at right angles upward to the



point indicated on the figure. Similarly $3-2i$, $-3+2i$, and $-3-2i$ may be located as on the figure.

Each complex number may be represented by one and only one point on the plane, and each point on the plane, not on an axis, locates one complex number.

Exercise 220

Graph each of the following complex numbers:

1. $1-2i$.	2. $-2+i$.	3. $-3+3i$.
4. $-1-2i$.	5. $-2-2i$.	6. $-1+i\sqrt{3}$.
7. $-\sqrt{2}-i\sqrt{3}$.	8. $\sqrt{3}+i\sqrt{2}$.	9. $-2-2i\sqrt{2}$.
10. $-1-2i\sqrt{2}$.	11. $2\sqrt{2}-i\sqrt{3}$.	12. $-\sqrt{3}+2i\sqrt{3}$.

232. Graphical addition and subtraction of complex numbers.

If the point A, $(3+i)$, and the point B, $(1+3i)$, are each connected with O and the parallelogram completed as in the

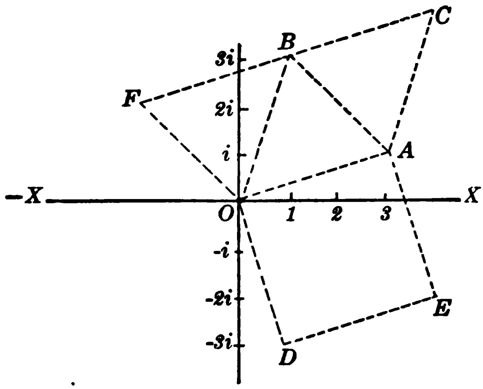


figure with A, O, and B as three consecutive vertices, then the fourth vertex, C, is the graph of the complex number $4+4i$, or the sum of $3+i$ and $1+3i$.

Similarly, the point E, $(4-2i)$, is the fourth vertex of the parallelogram of A, $(3+i)$, O, and D, $(1-3i)$.

The difference between two complex numbers may be represented as the fourth vertex of a parallelogram, one diagonal of which is the line formed by connecting the point of the minuend with O.

If we wish to find the point determined by $(1+3i)-(3+i)$, we complete the parallelogram O, A, $(3+i)$, and B, $(1+3i)$, which locates the point F, $(-2+2i)$.

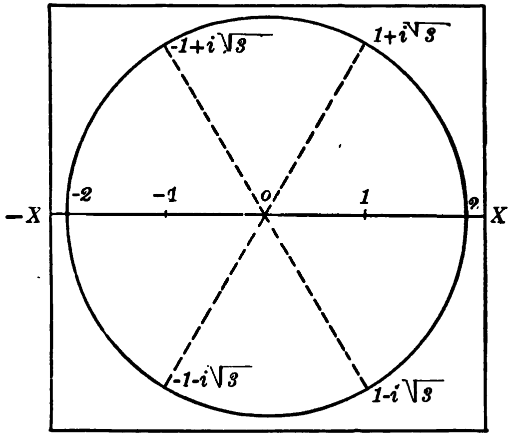
Exercise 221

Represent graphically the following:

1. $(3-2i)+(1+4i)$.
2. $(-2-3i)+(1-i)$.
3. $(-3+i)-(-2-2i)$.
4. $-(-2-2i)+(-2+2i)$.
5. $(4+4i)-(2-3i)$.
6. $(\sqrt{2}-2i)+(2-i\sqrt{3})$.
7. $(2\sqrt{2}-2i)-(2+2i\sqrt{3})$.
8. $-(2\sqrt{3}-i)+(\sqrt{3}-2i\sqrt{2})$.

233. Graphing roots of numbers. The point as the graph of a complex number furnishes a means for representing the roots of numbers.

The six sixth roots of 64 are located on the circumference of a circle, radius 2, and are 60° of arc apart, as in the figure.



Solution.

$x^6-64=0.$

Factoring, $(x-2)\ (x^2+2x+4)\ (x+2)\ (x^2-2x+4)=0.$

The two real roots are $x=2$ and $x=-2.$

Solving $x^2-2x+4=0,$

gives $x=1+\sqrt{-3}$, and $x=1-\sqrt{-3}.$

Solving $x^2+2x+4=0,$

gives $x=-1+\sqrt{-3}$, and $x=-1-\sqrt{-3}.$

These roots in the order of their location on the circle are $x=2$, $x=1+\sqrt{-3}$, $x=-1+\sqrt{-3}$, $x=-2$, $x=-1-\sqrt{-3}$, and $x=1-\sqrt{-3}.$

Similarly, locate the cube roots and the square roots of 64.

Exercise 222

Solve and locate all of the roots in each of the following:

1. $x^3-27=0$.
2. $x^4-16=0$.
3. $x^3+8=0$.
4. $x^6-729=0$.
5. $8x^3+27=0$.
6. $x^6+64=0$.

THE INDETERMINATE FORMS

234. Zero in multiplication. It is a rule of arithmetic that if one or more factors are zeros the product is zero; i.e., $7\cdot 0\cdot 5=0$. The same rule holds in algebra for $a\cdot b\cdot 0=0$.

235. Zero in division. Occasionally such forms as $\frac{0}{a}$, $\frac{a}{0}$, and $\frac{0}{0}$ are met with in algebra, especially in checking an equation or in evaluating a fraction. For instance, if, in evaluating the fraction $\frac{x+2}{x-2}$, we let $x=2$, the value of the fraction becomes $\frac{4}{0}$. If we let $x=-2$, its value becomes $\frac{0}{4}$. If, in evaluating the fraction $\frac{x^2-4}{x-2}$, we let $x=2$, its value becomes $\frac{0}{0}$.

That the form $\frac{0}{a}=0$ is evident if it is recalled that a product is zero only when a factor is zero, and that the dividend (the numerator of the fraction) is the product of the divisor (the denominator) and the quotient (the value of the fraction). Since the divisor is not zero the quotient must be. Therefore $\frac{0}{a}=0$.

The forms $\frac{a}{0}$ and $\frac{0}{0}$ can have no meaning in real numbers for division by zero cannot be allowed. But there are occasions when it is necessary to find some interpretation for $\frac{a}{0}$.

236. Interpretation of $\frac{a}{0}$. If 5 is divided by .1, the quotient is 50. If 5 is divided by .01 the quotient is 500, if by .001, the quotient is 5000, if by .0001, the quotient is 50000, etc. In equation form this becomes $\frac{5}{.1}=50$, $\frac{5}{.01}=500$, $\frac{5}{.001}=5000$, $\frac{5}{.0001}=50000$, . . .

Evidently, as the denominator decreases in size the value of the fraction increases, and, as the denominator becomes exceedingly small, the value of the fraction becomes exceedingly large.

In this series of operations the numerator, 5, remains unchanged throughout the discussion but the denominator changes. The numerator is said to be a **constant** and the denominator a **variable.**

A **constant** is a number that retains the same value throughout a particular mathematical discussion. A **variable** is a number that changes its value in the discussion.

The **limit of a variable** is that constant, the difference between which and the variable may be made to become and remain less than any assigned positive quantity, however small. The variable is said to approach this constant as its limit and the symbol $\doteq$ is used throughout mathematics to indicate this relation. Such an expression as $x \doteq a$ is to be read "x approaches a as its limit."

If the series of fractional equations $\frac{5}{.1}=50$, $\frac{5}{.01}=500$, $\frac{5}{.001}=5000$, $\frac{5}{.0001}=50000$, $\frac{5}{.00001}=500000$, . . . is continued indefinitely in the same manner, evidently the denominator of the left member becomes smaller and smaller and may be made less than any assigned quantity, however small, that is, the denominator $\doteq 0$. At the same time the value of the fraction

(the right member) will become larger and larger and will become and remain greater than any positive number which may be assigned. This necessitates a new definition and a new symbol.

If a number may become and remain greater than any positive number that may be assigned, it is said to become infinitely large or to become **infinite**; which means unbounded or unlimited. The symbol ∞ is called **infinity.** It is not the symbol for some number but it is the symbol that the value of the variable exceeds all bounds.

Now in interpreting $\frac{a}{0}$ if we replace $\frac{a}{0}$ by $\frac{a}{x}$ and consider the value of $\frac{a}{x}$ as $x \doteq 0$, evidently $\frac{a}{x}$ increases indefinitely and as $x \doteq 0$, $\frac{a}{x}$ becomes ∞.

Therefore $\frac{a}{0}$, where a is a constant, is said to have the value ∞ which is equivalent to the following:

Principle. *If the numerator of a fraction remains a constant, while the denominator approaches zero, the value of the fraction becomes* ∞.

The value of the fraction $\frac{x+2}{x-2}$ for $x=2$ is ∞.

237. Interpretation of $\frac{a}{\infty}$.

If we consider the series of fractions $\frac{5}{10}, \frac{5}{100}, \frac{5}{1000}, \frac{5}{10000}, \frac{5}{100000}, \ldots$, evidently, as the denominator increases indefinitely in the same manner, the value of the fraction decreases indefinitely.

Replacing $\frac{a}{\infty}$ by $\frac{a}{x}$ and assuming that $x \doteq \infty$ we have $\lim_{x \doteq \infty} \frac{a}{x} = 0$, which is to be read "the limit of $\frac{a}{x}$ as x approaches ∞ is zero."

Therefore $\frac{a}{\infty}$, where a is a constant, is said to have the value 0 which is equivalent to the following:

Principle. *If the numerator of a fraction remains a constant while the denominator approaches infinity, the value of the fraction approaches zero.*

An expression involving a single variable, that becomes indeterminate for a certain value of that variable, may be interpreted on the plan of the following **illustrative examples:**

1. Find the value of $\frac{x^2-4}{x-2}$ when $x=2$. Evidently, when $x=2$ the value of the given fraction is $\frac{0}{0}$.

But for x not equal to 2, $\frac{x^2-4}{x-2}=x+2$. Now, $\lim_{x \doteq 2} (x+2)=4$. Therefore, when $x=2$, we may give to $\frac{x^2-4}{x-2}$ the value 4.

2. Find the value of $\frac{4x^2+3x-1}{x^2+1}$ as $x \doteq \infty$.

For any finite value of x other than 0,

$$\frac{4x^2+3x-1}{x^2+1}=\frac{4+\frac{3}{x}-\frac{1}{x^2}}{1+\frac{1}{x^2}}.$$

$$\text{Now, } \lim_{x \doteq \infty} \left\{ \frac{4+\frac{3}{x}-\frac{1}{x^2}}{1+\frac{1}{x^2}} \right\} = \frac{4+0-0}{1+0}=4.$$

Therefore, when $x \doteq \infty$, give to $\frac{4x^2+3x-1}{x^2+1}$ the value 4.

Exercise 223

Find the value of each of the following as $x \doteq 0$:

1. $\dfrac{5}{x}$. Ans. ∞ (See principle).

2. $\dfrac{5}{x^2}$.

3. $\dfrac{5}{\frac{2}{x}}$. Ans. 0.

4. $\dfrac{x}{1-x}$.

5. $\dfrac{x^2}{2x}$.

6. $\dfrac{3x}{x-2}$.

Find the value of each of the following as $x \doteq \infty$:

7. $2x^3$.

8. $\dfrac{a}{x}$.

9. $2+\dfrac{2}{x}$.

10. $\dfrac{x}{x^2}$.

11. x^2+2.

12. $(x+2)^2$.

13. $\left(\dfrac{1}{x}+2\right)^2$.

Interpret each of the following:

14. $\underset{x \doteq 3}{\text{limit}} \left(\dfrac{x^2-9}{x^2-7x+12}\right)$. Ans. -6.

15. $\underset{x \doteq 2}{\text{limit}} \left(\dfrac{x^2+5x-14}{x^2-3x+2}\right)$.

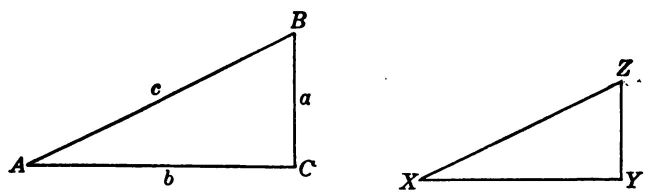
16. $\underset{x \doteq \infty}{\text{limit}} \left(\dfrac{x^2-2x+4}{x^2+2x+4}\right)$.

17. $\underset{x \doteq 5}{\text{limit}} \left(\dfrac{x^2+x-30}{x-5}\right)$.

18. $\underset{x \doteq \infty}{\text{limit}} \left(\dfrac{x+7}{x^2}\right)$.

TRIGONOMETRIC RATIOS

238. Trigonometry deals with the ratios obtained from the lengths of the sides of right triangles. The student will find when he comes to a more complete study of the subject that the development of its formulas requires a large use of algebra.



Given any two right triangles such as ACB and XYZ, as in the accompanying figures, with $\angle A = \angle X$, we know from a theorem of geometry that the two are mutually equiangular and therefore similar.

$$\therefore \frac{BC}{AB} = \frac{ZY}{XZ}, \ \frac{AC}{AB} = \frac{XY}{XZ}, \text{ and } \frac{BC}{AC} = \frac{ZY}{XY}. \quad \text{(Def.)}$$

Will these ratios remain equal for **all** rt. $\triangle$s having an acute $\angle$ equal to $\angle A$?

The constant ratio BC/AB is the **sine** of $\angle A$. That is, the **sine** of an acute angle of a right triangle is the ratio of the side opposite it and the hypotenuse.

The constant ratio AC/AB is the **cosine** of $\angle A$. That is, the **cosine** of an acute angle of a right triangle is the ratio of the side adjacent and the hypotenuse.

The constant ratio BC/AC is the **tangent** of $\angle A$. That is, the **tangent** of an acute angle of a right triangle is the ratio of the opposite and adjacent sides.

Using a, b, and c for the lengths of the three sides,

$$\sin A = \frac{a}{c}, \ \cos A = \frac{b}{c}, \ \tan A = \frac{a}{b}.$$

From the definitions it will be observed,

$$\sin B = \frac{b}{c}, \ \cos B = \frac{a}{c}, \ \tan B = \frac{b}{a}.$$

Exercise 224

1. Show that the sine of an acute angle is equal to the cosine of its complement.

2. Show that the sine of an acute angle is less than 1. Is this true for the cosine? The tangent?

3. If two acute angles are unequal which will have the greater sine? The greater cosine? The greater tangent?

4. What is the acute angle whose sine equals its cosine? What is the tangent of this angle?

5. Construct an equilateral triangle and one of its altitudes. Using the resulting right triangles, show

$$\sin 30° = \frac{1}{2} = .5,\ \cos 30° = \frac{\sqrt{3}}{2} = .866,\ \tan 30° = \frac{1}{\sqrt{3}} = .577,$$

$$\sin 60° = \frac{\sqrt{3}}{2} = .866,\ \cos 60° = \frac{1}{2} = .5,\ \tan 60° = \sqrt{3} = 1.732.$$

6. Construct an isosceles right triangle and show,
$\sin 45° = .707$, $\cos 45° = .707$, $\tan 45° = 1$.

Note. The trigonometric ratios on page 405 are correct to within .0005. They will be used in solving right triangles having given any side and either acute angle or any two sides. The lettering for the following exercises is the same as that for the first figure on page 402.

7. Given $A = 25°$, $c = 30$ in., find B, a, and b. Solution: $\sin A = a/c$, or $.423 = a/30$. Whence $a = 30(.423) = 12.69$ (in.) $B = 65°$ (Why?) $\sin B = b/c$. Whence $b = 27.18$ in.

8. Given $a = 15$ in., $b = 24$ in., find A, B, and c. Solution: $\tan A = a/b = 15/24 = .625$. By referring to the table we find the angle whose tangent is .625 is 32°. $\therefore A = 32°$ and $B = 58°$. Find c.

9. Given $B = 38°$, $a = 17$ in., find A, b, and c.

10. Given $b = 18$ ft., $c = 36$ ft., find A, B, and a.

11. Given $A = 23°$, $c = 75$ in., find B, a, and b.

12. Given $B = 43°$, $b = 6.23$ in., find A, a, and c.

13. When the sun is 40° high the shadow of a certain tree is 37 feet long. Find the height of the tree.

14. In order to determine the width of a river, a base line AC 100 feet long is measured along one bank. A point B is found on the opposite bank so that $\angle ACB$ is a right angle. If $\angle BAC$ is 73°, how wide is the river?

15. In order to determine the height of a tower CB, a base line CA is measured along the ground 150 feet long. The $\angle BAC$ is found to be 49°. How high is the tower?

Angle	sin	cos	tan	Angle	sin	cos	tan
0°	.000	1.000	.000	45°	.707	.707	1.000
1°	.017	1.000	.017	46°	.719	.695	1.036
2°	.035	.999	.035	47°	.731	.682	1.072
3°	.052	.999	.052	48°	.743	.669	1.111
4°	.070	.998	.070	49°	.755	.656	1.150
5°	.087	.996	.087	50°	.766	.643	1.192
6°	.105	.995	.105	51°	.777	.629	1.235
7°	.122	.993	.123	52°	.788	.616	1.280
8°	.139	.990	.141	53°	.799	.602	1.327
9°	.156	.988	.158	54°	.809	.588	1.376
10°	.174	.985	.176	55°	.819	.574	1.428
11°	.191	.982	.194	56°	.829	.559	1.483
12°	.208	.978	.213	57°	.839	.545	1.540
13°	.225	.974	.231	58°	.848	.530	1.600
14°	.242	.970	.249	59°	.857	.515	1.664
15°	.259	.966	.268	60°	.866	.500	1.732
16°	.276	.961	.287	61°	.875	.485	1.804
17°	.292	.956	.306	62°	.883	.469	1.881
18°	.309	.951	.325	63°	.891	.454	1.963
19°	.326	.946	.344	64°	.899	.438	2.050
20°	.342	.940	.364	65°	.906	.423	2.145
21°	.358	.934	.384	66°	.914	.407	2.246
22°	.375	.927	.404	67°	.921	.391	2.356
23°	.391	.921	.424	68°	.927	.375	2.475
24°	.407	.914	.445	69°	.934	.358	2.605
25°	.423	.906	.466	70°	.940	.342	2.747
26°	.438	.899	.488	71°	.946	.326	2.904
27°	.454	.891	.510	72°	.951	.309	3.078
28°	.469	.883	.532	73°	.956	.292	3.271
29°	.485	.875	.554	74°	.961	.276	3.487
30°	.500	.866	.577	75°	.966	.259	3.732
31°	.515	.857	.601	76°	.970	.242	4.011
32°	.530	.848	.625	77°	.974	.225	4.331
33°	.545	.839	.649	78°	.978	.208	4.705
34°	.559	.829	.675	79°	.982	.191	5.145
35°	.574	.819	.700	80°	.985	.174	5.671
36°	.588	.809	.727	81°	.988	.156	6.314
37°	.602	.799	.754	82°	.990	.139	7.115
38°	.616	.788	.781	83°	.993	.122	8.144
39°	.629	.777	.810	84°	.995	.105	9.514
40°	.643	.766	.839	85°	.996	.087	11.430
41°	.656	.755	.869	86°	.9976	.070	14.301
42°	.669	.743	.900	87°	.9986	.052	19.081
43°	.682	.731	.933	88°	.9994	.035	28.636
44°	.695	.719	.966	89°	.9998	.017	57.290
45°	.707	.707	1.000	90°	1.000	.000	......

No.	Square	Cube	Square Root	Cube Root	No.	Square	Cube	Square Root	Cube Root
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260–	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.780–
5	25	125	2.236	1.710	55	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.826–
7	49	343	2.646–	1.913–	57	3,249	185,193	7.550–	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.616–	3.871–
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	60	3,600	216,000	7.746	3.915–
11	121	1,331	3.317–	2.224	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.958–
13	169	2,197	3.606–	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.742–	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	65	4,225	274,625	8.062	4.021–
16	256	4,096	4.000	2.520–	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061–
18	324	5,832	4.243–	2.621–	68	4,624	314,432	8.246	4.082–
19	361	6,859	4.359–	2.668	69	4,761	328,509	8.307–	4.102
20	400	8,000	4.472	2.714	70	4,900	343,000	8.367–	4.121
21	441	9,261	4.583–	2.759–	71	5,041	357,911	8.426	4.141–
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.796–	2.844–	73	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.718–	4.236–
27	729	19,683	5.196	3.000	77	5,929	456,533	8.775	4.254
28	784	21,952	5.291	3.037–	78	6,084	474,552	8.832–	4.273–
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.309–
31	961	29,791	5.568–	3.141	81	6,561	531,441	9.000	4.327–
32	1,024	32,768	5.657–	3.175–	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745–	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240–	84	7,056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.397–
36	1,296	46,656	6.000	3.302–	86	7,396	636,056	9.274–	4.414
37	1,369	50,653	6.083–	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	88	7,744	681,472	9.381–	4.448
39	1,521	59,319	6.245	3.391	89	7,921	704,969	9.434	4.465–
40	1,600	64,000	6.325	3.420	90	8,100	729,000	9.487–	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.498–
42	1,764	74,088	6.481–	3.476	92	8,464	778,688	9.592–	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.644–	4.531–
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.547–
45	2,025	91,125	6.708	3.557–	95	9,025	857,375	9.747–	4.563–
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.798	4.579–
47	2,209	103,823	6.856–	3.609–	97	9,409	912,673	9.849–	4.595–
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.950–	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.642–

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Newell, M.J.
High school algebra.

NAME	DATE	NAME	DATE